

QUESTION 1

The blanks below will be filled by students. (Except the score)

Surname:	Name: KEY	Group Number:	List Number:	Score
Signature:	Electronic Post(e-mail) address:	Student Number:		

For the solution of this question please use only the front face and if necessary the back face of this page.

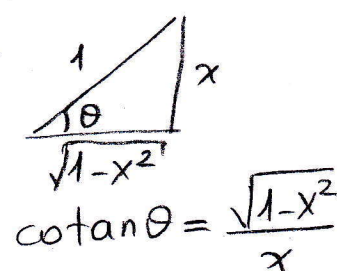
[10 pts] a) Evaluate the integral $\int \frac{dx}{x^2 \sqrt{1-x^2}}$ using trigonometric substitution.

[10 pts] b) Evaluate the integral $\int \frac{x}{(x+2)^2(x+1)} dx$.

[5 pts] c) Find $\frac{dy}{dx}$ if $y = x^x$.

a)
$$\int \frac{dx}{x^2 \sqrt{1-x^2}} = \int \frac{\cos \theta}{\sin^2 \theta \sqrt{1-\sin^2 \theta}} d\theta$$

$x = \sin \theta$
 $dx = \cos \theta d\theta$



$$= \int \frac{\cancel{\cos \theta}}{\sin^2 \theta \cancel{\cos \theta}} d\theta$$

$$= \int \frac{d\theta}{\sin^2 \theta} = \int \operatorname{cosec}^2 \theta d\theta$$

$$= -\cot \theta + C = -\frac{\sqrt{1-x^2}}{x} + C$$

b)
$$\frac{x}{(x+2)^2(x+1)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x+1}$$

$$x = A(x+2)(x+1) + B(x+1) + C(x+2)^2$$

$$\left. \begin{aligned} x = -2 &\Rightarrow -2 = -B \\ x = -1 &\Rightarrow -1 = C \\ x = 0 &\Rightarrow 0 = 2A + B + 4C \end{aligned} \right\} \begin{aligned} A &= 1 \\ B &= 2 \\ C &= -1 \end{aligned}$$

$$\int \frac{x}{(x+2)^2(x+1)} dx = \int \frac{dx}{x+2} + \int \frac{2}{(x+2)^2} dx - \int \frac{dx}{x+1}$$

$$= \ln|x+2| - \frac{2}{x+2} - \ln|x+1| + C$$

c) $y = x^x$
 $\ln y = \ln(x^x)$
 $\ln y = x \ln x$

Taking the derivative of both sides of equation we have

$$\frac{y'}{y} = 1 \cdot \ln x + x \cdot \frac{1}{x} \Rightarrow y' = x^x (1 + \ln x)$$

QUESTION 2

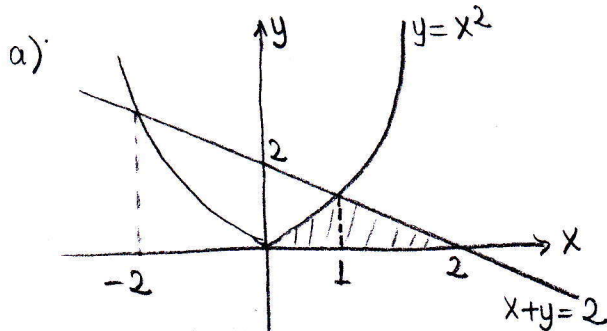
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[10 pts] a) Find the area of the region **R** enclosed by the curve $y = x^2$ and the line $x + y = 2$ and the x -axis by using definite integrals.

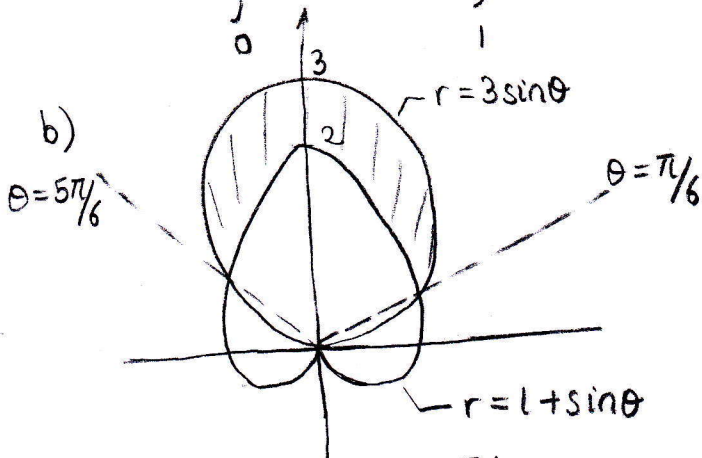
[15 pts] b) Find the area of the region **R** inside the circle $r = 3 \sin \theta$ and outside the cardioid $r = 1 + \sin \theta$.



$$x^2 = 2 - x \Rightarrow x^2 + x - 2 = 0$$

$$x = -2 \quad x = 1 \quad \text{intersection points.}$$

$$\text{Area} = \int_0^1 x^2 dx + \int_1^2 (2-x) dx = \left. \frac{x^3}{3} \right|_0^1 + \left. \left(2x - \frac{x^2}{2} \right) \right|_1^2 = \frac{1}{3} + \left(2 - \frac{3}{2} \right) = \frac{5}{6}$$



$$1 + \sin \theta = 3 \sin \theta$$

$$\sin \theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\text{Area} = 2 \left[\frac{1}{2} \int_{\pi/6}^{\pi/2} (3 \sin \theta)^2 d\theta - \frac{1}{2} \int_{\pi/6}^{\pi/2} (1 + \sin \theta)^2 d\theta \right]$$

$$= \int_{\pi/6}^{\pi/2} (8 \sin^2 \theta - 1 - 2 \sin \theta) d\theta$$

$$= 8 \int_{\pi/6}^{\pi/2} \frac{1 - \cos(2\theta)}{2} d\theta + (-\theta + 2 \cos \theta) \Big|_{\pi/6}^{\pi/2}$$

$$= 4 \left[\theta - \frac{\sin(2\theta)}{2} \right] \Big|_{\pi/6}^{\pi/2} + \left(-\frac{\pi}{2} + \frac{\pi}{6} - \sqrt{3} \right)$$

$$= 4 \left(\frac{\pi}{2} - \frac{\pi}{6} + \frac{\sqrt{3}}{2} \right) + \left(-\frac{\pi}{2} - \sqrt{3} \right) = \pi$$

QUESTION 3

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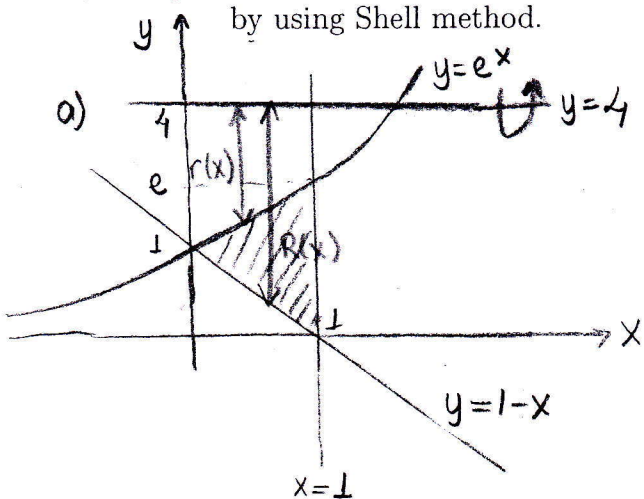
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R is the region enclosed by the curve $y = e^x$ and the lines $y = 1 - x$ and $x = 1$.

[10 pts] i) Write the definite integral that calculates the volume of the solid generated by revolving the region R about the line $y = 4$ by using Disk/Washer method.

(Do not evaluate the definite integral.)

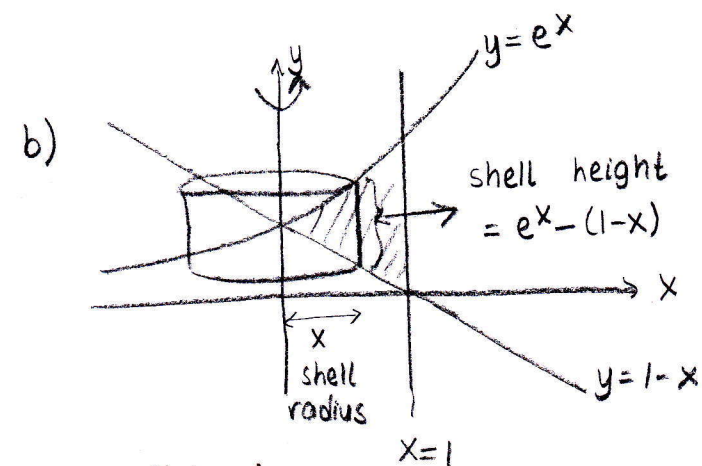
[15 pts] ii) Find the the volume of the solid generated by revolving the region R about the y -axis by using Shell method.



a) $R(x) = 4 - (1-x) = 3+x$
 $r(x) = 4 - e^x$

$$V_{y=4} = \pi \int_0^1 \left\{ [R(x)]^2 - [r(x)]^2 \right\} dx$$

$$= \pi \int_0^1 \left[(4-(1-x))^2 - (4-e^x)^2 \right] dx$$



$$V_y = 2\pi \int_0^1 (\text{shell radius}) (\text{shell height}) dx$$

$$= 2\pi \int_0^1 x [e^x - (1-x)] dx$$

$$= 2\pi \int_0^1 x (e^x + x - 1) dx$$

$x=u \quad e^x dx = dv$
 $dx = du \quad v = e^x$

$$= 2\pi \left[\int_0^1 x e^x dx + \int_0^1 (x^2 - x) dx \right]$$

$$= 2\pi \left[x e^x \Big|_0^1 - \int_0^1 e^x dx + \left(\frac{x^3}{3} - \frac{x^2}{2} \right) \Big|_0^1 \right] = 2\pi \cdot \frac{5}{6} = \frac{5\pi}{3}$$

QUESTION 4

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[8 pts] a) $\lim_{x \rightarrow 0} \frac{\int_0^{\sinh x} e^{t^2} dt}{2^x - e^{\sin x}} = ?$

[7+10 pts] b) Investigate the convergence or divergence of the following improper integrals.

i) $\int_1^{\infty} \frac{1 + \sin^2 x}{\sqrt{x}} dx$

ii) $\int_1^2 \frac{dx}{\sqrt{4-x^2}}$

a) $\lim_{x \rightarrow 0} \frac{\int_0^{\sinh x} e^{t^2} dt}{2^x - e^{\sin x}} = \left(\frac{0}{0}\right) \lim_{x \rightarrow 0} \frac{e^{(\sinh x)^2} \cosh x}{2^x \cdot \ln 2 - e^{\sin x} \cos x} = \frac{1}{\ln 2 - 1}$

b) i) $0 < \frac{1}{\sqrt{x}} \leq \frac{1 + \sin^2 x}{\sqrt{x}} \quad x \in [1, \infty)$

$\int_1^{\infty} \frac{dx}{\sqrt{x}}$ is divergent since $p = \frac{1}{2} \leq 1$. By the Direct

Comparison test, $\int_1^{\infty} \frac{1 + \sin^2 x}{\sqrt{x}} dx$ is also divergent.

ii) $\int_1^2 \frac{dx}{\sqrt{4-x^2}} = \lim_{b \rightarrow 2^-} \int_1^b \frac{dx}{\sqrt{4-x^2}}$

$= \lim_{b \rightarrow 2^-} \sin^{-1}\left(\frac{x}{2}\right) \Big|_1^b$

$= \lim_{b \rightarrow 2^-} \left[\sin^{-1}\left(\frac{b}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right) \right]$

$= \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$

Since the limit is finite, $\int_1^2 \frac{dx}{\sqrt{4-x^2}}$ is convergent.