MATHEMATICS 101E FINAL JANUARY 9, 2012

QUESTION 1 The blanks below will be filled by students. (Except the score)

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Evaluate the following integrals.

[13 pt] a) $\displaystyle\int_{\frac{1}{2}}^{1} \frac{\sin^{-1} x}{x^3} dx,$ (Note: $\sin^{-1} x = \arcsin x$)

[12 pt] b) $\displaystyle\int \frac{\sqrt[3]{x} - 1}{\sqrt[3]{x^4}(\sqrt[3]{x^2} + 1)} dx$ by using the substitution $x = t^3$.

a) $\displaystyle\int_{\frac{1}{2}}^{1} \frac{\sin^{-1} x}{x^3} dx$, $\left[\sin^{-1} x = t \Rightarrow x = \sin t \quad dx = \cos t\,dt\right]$ $\left[\begin{array}{l} x = 1/2 \Rightarrow t = \pi/6 \\ x = 1 \Rightarrow t = \frac{\pi}{2}\end{array}\right]$

$$= \int_{\pi/6}^{\pi/2} \frac{t}{\sin^3 t} \cdot \cos t\, dt \qquad \left[\begin{array}{ll} u = t & dv = \frac{\cos t\,dt}{\sin^3 t} \\ du = dt & v = -\frac{1}{2\sin^2 t}\end{array}\right]$$

$$= -\frac{t}{2\sin^2 t}\Bigg|_{\pi/6}^{\pi/2} + \int_{\pi/6}^{\pi/2} \frac{1}{2\sin^2 t} dt = -\frac{1}{2}\left[\frac{\pi}{2.1} - \frac{\pi}{6.\frac{1}{4}}\right] + \frac{1}{2}\int_{\pi/6}^{\pi/2} \operatorname{cosec}^2 t\, dt$$

$$= \frac{\pi}{12} - \frac{1}{2}\cot t\Bigg|_{\pi/6}^{\pi/2} = \frac{\pi}{12} - \frac{1}{2}\left(0 - \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) = \frac{\pi}{2} + \frac{\sqrt{3}}{2}$$

b) $\displaystyle\int \frac{\sqrt[3]{x} - 1}{\sqrt[3]{x^4}(\sqrt[3]{x^2}+1)} dx$, $\left[x = t^3,\ dx = 3t^2 dt\right]$

$$= \int \frac{(t-1)}{t^4(t^2+1)} 3t^2 dt = 3\int \frac{(t-1)}{t^2(t^2+1)} dt = 3\left[\int \frac{dt}{t} - \int \frac{dt}{t^2} - \int \frac{t-1}{t^2+1} dt\right]$$

$$\left[\begin{array}{l} \frac{t-1}{t^2(t^2+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{Ct+D}{t^2+1} \\ t-1 = At(t^2+1) + B(t^2+1) + (Ct+D)t^2 \\ \quad = t^3(A+C) + t^2(B+D) + At + B \\ A = 1, B = -1, C = -1, D = 1 \end{array}\right]$$

$$= 3\left\{\ln|t| + \frac{1}{t} - \frac{1}{2}\ln(t^2+1) + \tan^{-1} t + c\right\}$$

$$= 3\ln|x^{1/3}| + \frac{3}{x^{1/3}} - \frac{3}{2}\ln(x^{2/3}+1) + 3\tan^{-1}(x^{1/3}) + C$$

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QUESTION 2 The blanks below will be filled by students. (Except the score)

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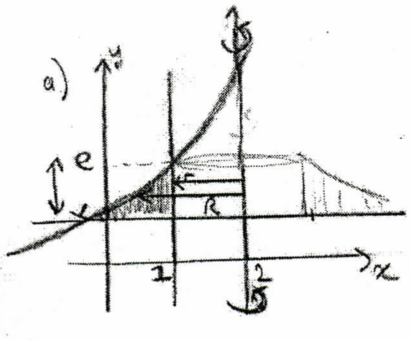
[17 pt] a) The region enclosed by the curve $\mathbf{y = e^x}$ and the lines $\mathbf{x = 1}$ and $\mathbf{y = 1}$ is revolved about the line $\mathbf{x = 2}$ to generate a solid. Set up the definite integrals that calculate the volume
i) by using the Washer Method,
ii) by using the Shell Method.
iii) Calculate the volume of the solid by evaluating one of the integrals obtained in i) and ii).

[8 pt] b) Find the length of the curve given by the parametric equations $\mathbf{x(t) = \cosh 2t}$, $\mathbf{y(t) = 2t}$ for $\mathbf{0 \le t \le \ln 2}$.

a)



i) The Washer method:

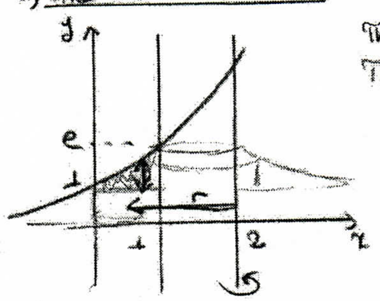
The outer radius : $y = e^x \Rightarrow x = \ln y$, $R(y) = 2 - \ln y$

The inner radius : $x = 1 \Rightarrow r(y) = 1$

The limits of the integral : from $y = 1$ to $y = e$

$$V = \int_\alpha^\beta \pi \left[R^2(y) - r^2(y)\right] dy = \pi \int_1^e \left[(2-\ln y)^2 - 1^2\right] dy \quad //$$

ii) The Shell method:



The shell radius: $2 - x$
The shell height: $e^x - 1$

$$\left.\right\} V = 2\pi \int_{x=0}^{1} \left(\text{shell radius}\right)\left(\text{shell height}\right)\left(\text{shell thickness}\right)$$

$$V = 2\pi \int_0^1 (2-x).(e^x - 1)dx \quad //$$

iii) by the washer method;

iii) by the shell method:

$$V = 2\pi \left\{ \int_0^1 (2-x)e^x dx - \int_0^1 (2-x)dx \right\}$$

$$\left[\begin{matrix} u = (2-x) & dv = e^x dx \\ du = -dx & v = e^x \end{matrix}\right] \Rightarrow V = 2\pi \left[(2-x)e^x + e^x - 2x + \frac{x^2}{2} \right]_0^1$$

$$V = 2\pi \left\{ \left[e + e - 2 + \tfrac{1}{2}\right] - \left[2 + 1\right] \right\} = 2\pi\left(2e - \tfrac{9}{2}\right) \quad //$$

b) $$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{\ln 2} \sqrt{4\sinh^2 2t + 4}\, dt = 2\int_0^{\ln 2} \sqrt{\sinh^2 2t + 1}\, dt, \quad \cosh^2 t - \sinh^2 t = 1$$

$$\left.\begin{matrix} x = \cosh 2t \\ y = 2t \end{matrix} \Rightarrow \begin{matrix} \frac{dx}{dt} = 2\sinh 2t \\ \frac{dy}{dt} = 2 \end{matrix}\right|$$

$$= 2\int_0^{\ln 2} \sqrt{\cosh^2 2t}\, dt = 2\int_0^{\ln 2} \cosh 2t\, dt = 2 \cdot \tfrac{1}{2} \sinh 2t \Big|_0^{\ln 2} = \sinh(2\ln 2)$$

$$= \left\{ \frac{e^{2\ln 2} - e^{-2\ln 2}}{2} \right\} = \frac{1}{2}\left(4 - \frac{1}{4}\right) = \frac{15}{8} \quad //$$

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QUESTION 3 The blanks below will be filled by students. (Except the score)

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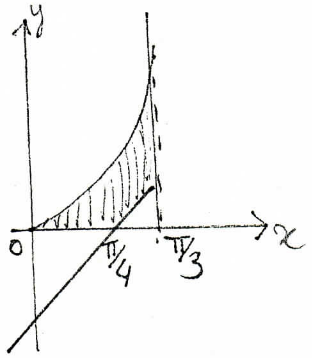
[12 pt] a) Find the area of the region in the first quadrant bounded on the right by the line $\mathbf{x} = \frac{\pi}{3}$, below by the x axis and the line $\mathbf{y} = \mathbf{x} - \frac{\pi}{4}$, above by the curve $\mathbf{y} = \tan \mathbf{x}$.

[13 pt] b) Find the area of the region inside the circle $\mathbf{r} = \mathbf{3}$ and outside the cardioid $\mathbf{r} = \mathbf{2(1 - \cos\theta)}$.

a)

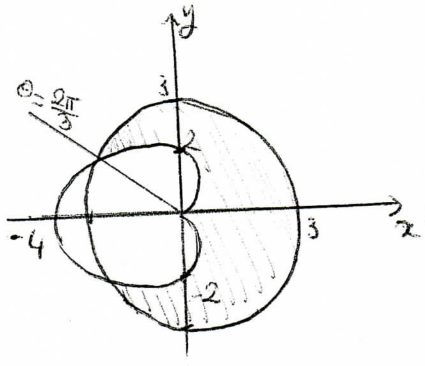


$$\text{Area} = \int_0^{\pi/3} \tan x\,dx - \int_{\pi/4}^{\pi/3} \left(x - \frac{\pi}{4}\right) dx$$

$$= -\ln|\cos x| \Big|_0^{\pi/3} - \left[\frac{x^2}{2} - \frac{\pi}{4}x\right]_{\pi/4}^{\pi/3} = -\ln\frac{1}{2} + \ln 1 - \left\{\left[\frac{\pi^2}{18} - \frac{\pi^2}{12}\right] - \left[\frac{\pi^2}{32} - \frac{\pi^2}{16}\right]\right\}$$

$$= \ln 2 - \frac{\pi^2}{288} \quad //$$

b)



The intersection points; $r_1 = 3$, $r_2 = 2(1-\cos\theta)$ $\frac{2\pi}{3}$ $\frac{4\pi}{3}$ $-\frac{2\pi}{3}$

$r_1 = r_2 \Rightarrow 3 = 2 - 2\cos\theta \Rightarrow \cos\theta = -\frac{1}{2}$, $\theta = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}; -\pi + \frac{\pi}{3}, \ldots$

$$\text{Area} = \frac{1}{2}\int_\alpha^\beta [r_1(\theta)^2 - r_2(\theta)^2]\,d\theta$$

$$\text{Area} = 2 \cdot \frac{1}{2}\int_0^{2\pi/3} [3^2 - (2 - 2\cos\theta)^2]\,d\theta$$

$$= \int_0^{2\pi/3} [9 - 4 + 8\cos\theta - 4\cos^2\theta]\,d\theta = \int_0^{2\pi/3} [5 + 8\cos\theta - 2(1 + \cos 2\theta)]\,d\theta$$

$$= \int_0^{2\pi/3} [3 + 8\cos\theta - 2\cos 2\theta]\,d\theta = [3\theta + 8\sin\theta - \sin 2\theta]\Big|_0^{2\pi/3} = \left[\frac{3 \cdot 2\pi}{3} + 8\overbrace{\sin\frac{2\pi}{3}}^{\sqrt{3}/2} - \overbrace{\sin\frac{4\pi}{3}}^{-\sqrt{3}/2}\right] - 0 = 2\pi + \frac{9\sqrt{3}}{2}$$

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QUESTION 4 The blanks below will be filled by students. (Except the score)

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[10 pt] a) Evaluate the limit $\lim\limits_{x\to 0^+} (\frac{\sin x}{x})^{\frac{1}{x}} = ?$

[8 pt] b) For what value of β does the integral $\int_1^\infty (\frac{\beta}{x+1} - \frac{1}{x})dx$ converge? Evaluate the corresponding integral for this value of β.

[7 pt] c) Determine the convergence of the integral $\int_2^\infty \frac{x+\cos^2 x}{x^3}dx$.

a) $\lim\limits_{x\to 0^+} \left(\frac{\sin x}{x}\right)^{1/x} \overset{(1^\infty)}{=} e^{\lim\limits_{x\to 0^+} \ln\left(\frac{\sin x}{x}\right)^{1/x}} = e^{\boxed{\lim\limits_{x\to 0^+} \frac{1}{x}\ln\left(\frac{\sin x}{x}\right)}A}$

For A: $A = \lim\limits_{x\to 0^+} \frac{1}{x}\ln\left(\frac{\sin x}{x}\right)^{\infty . 0} = \lim\limits_{x\to 0^+} \frac{\ln\left(\frac{\sin x}{x}\right)}{x} \overset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to 0^+} \frac{\frac{(x\cos x - \sin x)}{x^2} \Big/ \frac{\sin x}{x}}{1}$

$= \lim\limits_{x\to 0^+} \frac{x\cos x - \sin x}{x\sin x} \overset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to 0^+} \frac{\cancel{\cos x} - x\sin x - \cancel{\cos x}}{\sin x + x\cos x} =$

$\Rightarrow \lim\limits_{x\to 0^+} \frac{-x\sin x}{x(\underbrace{\frac{\sin x}{x}}_{1} + \underbrace{\cos x}_{1})} = 0 \Rightarrow \lim\limits_{x\to 0^+} \left(\frac{\sin x}{x}\right)^{1/x} = e^0 = 1$

b) i) $\int_1^\infty \left(\frac{\beta}{x+1} - \frac{1}{x}\right)dx = \lim\limits_{b\to\infty} \int_1^b \left(\frac{\beta}{x+1} - \frac{1}{x}\right)dx = \lim\limits_{b\to\infty} \Big[\beta \ln|x+1| - \ln|x|\Big]_1^b$

$= \lim\limits_{b\to\infty} \ln\left|\frac{(x+1)^\beta}{x}\right| \Big]_1^b = \lim\limits_{b\to\infty} \left[\ln\left|\frac{(b+1)^\beta}{b}\right| - \ln 2^\beta\right]$

If $\beta = 1$, then the limit exist, and the improper integral converges.

ii) $\int_1^\infty \left(\frac{1}{x+1} - \frac{1}{x}\right)dx = \lim\limits_{b\to\infty} \ln\left|\frac{b+1}{b}\right| - \ln 2 = -\ln 2$

c) $\int_2^\infty \underbrace{\frac{x+\cos^2 x}{x^3}}_{f(x)} dx$ (type 1), $f(x) = \frac{x+\cos^2 x}{x^3} \leq \frac{x+1}{x^3} < \frac{2x}{x^3} = \frac{2}{x^2} = g(x)$, $x>2$

Since $\int_2^\infty g(x)dx = 2\int_2^\infty \frac{dx}{x^2}$ $(p=2>1)$ converges and $0 < f(x) < g(x)$,

$\int_2^\infty \frac{x+\cos^2 x}{x^3}dx$ converges (by the direct comparison test)