MATHEMATICS 101-101E103-103E — FINAL EXAM KEY — 23.05.2013

QUESTION 1 The blanks below will be filled by students. (Except the score)

Surname:	Name:	Group Number:	List Number:	Score
Signature:	Electronic Post(e-mail) address:	Student Number:		

For the solution of this question please use only the front face and if necessary the back face of this page.

[10 pts] a) Evaluate the integral $\int \frac{dx}{(x^2-1)^{3/2}}$ by using trigonometric substitution.

[10 pts] b) Evaluate the integral $\int \frac{x^2+5x+2}{(x+1)(x^2+1)}\, dx$.

[5 pts] c) Find $\frac{d}{dx}\int_{\sqrt{x}}^{0} \sin(t^2)\, dt$.

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a) $x=\sec t \qquad dx = \sec t . \tan t\, dt \Rightarrow (x^2-1)^{3/2} = (\sec^2 t - 1)^{3/2} = (\tan^2 t)^{3/2}$

$$\int \frac{dx}{(x^2-1)^{3/2}} = \int \frac{\sec t . \cancel{\tan t}\, dt}{\tan^{\cancel{3}\,2} t} = \int \frac{1}{\cos t}\cdot\frac{\cos^2 t}{\sin^2 t}\, dt = \int \frac{\cos t}{\sin^2 t}\, dt$$

$$\left.\begin{array}{l} u=\sin t \\ du=\cos t\, dt \end{array}\right\} \Rightarrow \int \frac{dx}{(x^2-1)^{3/2}} = \int \frac{du}{u^2} = -\frac{1}{u} + C' = -\frac{1}{\sin t} + C = -\frac{x}{\sqrt{x^2-1}} + C$$

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b) $$\frac{x^2+5x+2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$\Rightarrow A(x^2+1) + (Bx+C)(x+1) = x^2+5x+2 \Rightarrow Ax^2 + A + Bx^2 + Bx + Cx + C = x^2+5x$

$\Rightarrow (A+B)x^2 + (B+C)x + A + C = x^2+5x+2$

$$\Rightarrow \left.\begin{array}{l} A+B=1 \\ B+C=5 \\ A+C=2 \end{array}\right\} \Rightarrow 2A+2B+2C=8 \Rightarrow A+B+C=4 \Rightarrow A=-1,\ B=2,\ C=3$$

$$\frac{x^2+5x+2}{(x+1)(x^2+1)} = \frac{-1}{x+1} + \frac{2x+3}{x^2+1}$$

$u=x^2+1$
$du=2x\,dx$

$$\int \frac{x^2+5x+2}{(x+1)(x^2+1)}\, dx = \int \frac{-dx}{x+1} + \int \frac{2x+3}{x^2+1}\, dx = -\int \frac{dx}{x+1} + \int \frac{2x\,dx}{x^2+1} + 3\int \frac{dx}{x^2+1}$$

$$= -\ln|x+1| + \ln(x^2+1) + 3\tan^{-1} x + C$$

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c) $$\frac{d}{dx}\int_{\sqrt{x}}^{0} \sin(t^2)\, dt = \frac{d}{dx}\left(-\int_{0}^{\sqrt{x}} \sin(t^2)\, dt\right) \underset{\uparrow}{=} -\sin(\sqrt{x})^2 \cdot \frac{1}{2\sqrt{x}} = -\frac{\sin x}{2\sqrt{x}}$$

By the FTC

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QUESTION 2 The blanks below will be filled by students. (Except the score)

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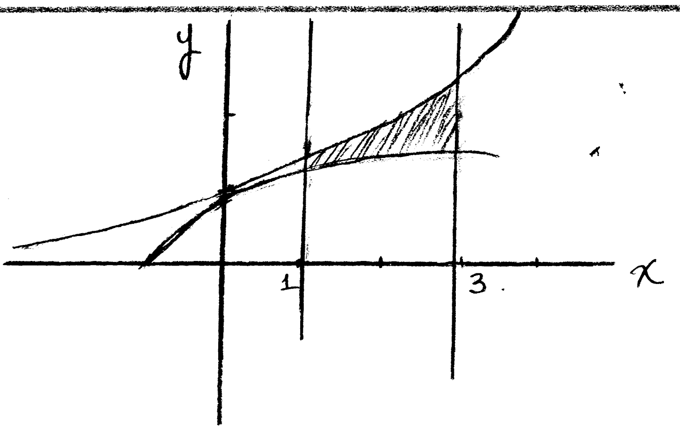
[7 pts] a) Find the area of the region $\mathbf{R}$ in the first quadrant enclosed above by $\mathbf{y = e^x}$ and below by $\mathbf{y = \sqrt{x+1}}$ where $\mathbf{1 \le x \le 3}$ using definite integral.

[13 pts] b) Find the area of the region $\mathbf{R}$ inside the circle $\mathbf{r = 3\cos\theta}$ and outside the cardioid $\mathbf{r = 1 + \cos\theta}$.

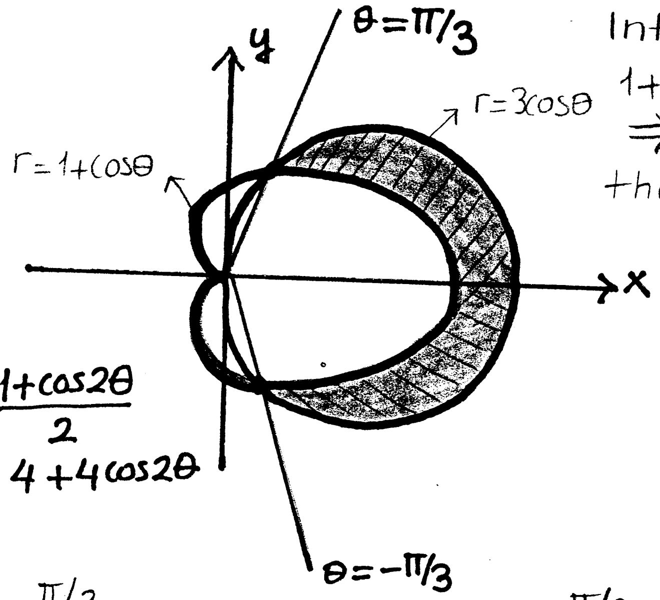
a)



$$A = \int_1^3 \left(e^x - \sqrt{x+1}\right)dx$$

$$A = \left[e^x - \frac{2}{3}(x+1)^{3/2}\right]\Bigg|_1^3 = e^3 - \frac{2}{3}4^{3/2} - e + \frac{2}{3}2^{3/2} = e^3 - e + \frac{4\sqrt{2}-16}{3}$$

b)



Intersection points:

$1+\cos\theta = 3\cos\theta \Rightarrow 2\cos\theta = 1 \Rightarrow \cos\theta = 1/2$

$\Rightarrow \theta = \pm\pi/3$ (The graph also gives the point of intersection (0,0)).

$$A = 2\int_0^{\pi/3} \frac{1}{2}\left[(3\cos\theta)^2 - (1+\cos\theta)^2\right]d\theta$$

$$A = \int_0^{\pi/3} \left(9\cos^2\theta - 1 - 2\cos\theta - \cos^2\theta\right)d\theta$$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$8\cos^2\theta = 4 + 4\cos 2\theta$

$$A = \int_0^{\pi/3} \left(8\cos^2\theta - 2\cos\theta - 1\right)d\theta = \int_0^{\pi/3} \left(4\cos 2\theta - 2\cos\theta + 3\right)d\theta$$

$$A = \left(2\sin 2\theta - 2\sin\theta + 3\theta\right)\Big|_0^{\pi/3} = 2\sin\frac{2\pi}{3} - 2\sin\frac{\pi}{3} + 3\cdot\frac{\pi}{3} = \pi$$

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QUESTION 3 The blanks below will be filled by students. (Except the score)

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[10 pts] **a)** Find the length of the curve $\mathbf{y} = \ln(\cos \mathbf{x})$ where $\mathbf{0} \le \mathbf{x} \le \frac{\pi}{3}$.

[15 pts] **b)** Let **R** be the region bounded above by the curve $\mathbf{y} = \cosh \mathbf{x}$, below by the curve $\mathbf{y} = \sinh \mathbf{x}$, from the left by the y-axis and from the right by the line $\mathbf{x} = \mathbf{1}$.

i. Draw the region **R**. (**2 pts**)

ii. Find the volume of the solid generated by revolving the region **R** about the **x**-axis using **Disk/Washer** method. (**8 pts**)

iii. Write the definite integral that calculates the volume of the solid generated by revolving the region **R** about **y**-axis using **Shell** method. Do not evaluate the integral. (**5 pts**)

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a) $y = \ln(\cos x)$ $\quad \frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$

$$L = \int_0^{\pi/3} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{\pi/3} \sqrt{1+\tan^2 x}\, dx = \int_0^{\pi/3} |\sec x|\, dx = \int_0^{\pi/3} \sec x\, dx$$

$$= \int_0^{\pi/3} \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\, dx = \int_0^{\pi/3} \frac{\sec^2 x + \sec x \cdot \tan x}{\sec x + \tan x}\, dx$$

$$= \ln|\sec x + \tan x| \Big|_0^{\pi/3} = \ln|\sec \pi/3 + \tan \pi/3| - \ln|\sec 0 + \tan 0|$$

$$= \ln(2+\sqrt{3}) - \ln 1 = \ln(2+\sqrt{3})$$

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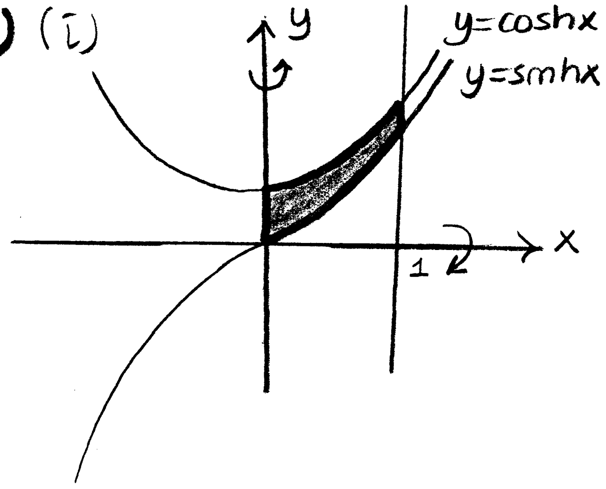
b) (i)



$\cosh^2 x - \sinh^2 x = 1$

(ii) $V_{y=0} = \pi \int_0^1 (\underbrace{\cosh^2 x - \sinh^2 x}_{=1})\, dx$

$$= \pi \int_0^1 dx = \pi$$

(iii) $V_{x=0} = 2\pi \int \underbrace{x}_{r} \underbrace{(\cosh x - \sinh x)}_{h}\, d$

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QUESTION 4 The blanks below will be filled by students. (Except the score)

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[12 pts] a) Investigate the convergence or divergence of the improper integral $\int_0^1 x\ln x\,dx$ **by evaluating the integral.**

[8 pts] b) Investigate the convergence or divergence of the improper integral $\int_3^\infty \frac{1}{x-e^{-x}}dx$ **by using a convenient test.**

[10 pts] c) Find the limit $\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$.

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a) $\int x\ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\frac{dx}{x} = \frac{x^2}{2}\ln x - \frac{x^2}{4}+C$

$\left\{\begin{array}{ll} u=\ln x & du=\frac{dx}{x} \\ dv = x\,dx & v=\frac{x^2}{2}\end{array}\right\}$

$\int_0^1 x\ln x\,dx = \lim_{a\to 0^+}\int_a^1 x\ln x\,dx = \lim_{a\to 0^+}\frac{1}{4}\left(2x^2\ln x - x^2\right)\Big|_a^1$

$= \frac{1}{4}\lim_{a\to 0^+}\left[(2\ln 1-1)-(2a^2\ln a - a^2)\right] = \frac{1}{4}\lim_{a\to 0^+}\left[a^2(1-2\ln a)-1\right]$

$\lim_{a\to 0^+}\left[a^2(1-2\ln a)\right] \overset{0\cdot\infty}{=} \lim_{a\to 0^+}\frac{1-2\ln a}{\frac{1}{a^2}} \overset{\frac{\infty}{\infty}}{=} \lim_{a\to 0^+}\frac{-\frac{2}{a}}{-\frac{2}{a^3}} = \lim_{a\to 0^+}\frac{2}{a}\cdot\frac{a^3}{2}=0$

L'Hôpital's Rule

$\int_0^1 x\ln x\,dx = \frac{1}{4}\lim_{a\to 0^+}\left[a^2(1-2\ln a)-1\right] = \frac{1}{4}(0-1) = -\frac{1}{4}\in\mathbb{R}.$

Therefore, the integral converges.

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b) $x-e^{-x} = x-\frac{1}{e^x}$ is defined $(\forall x\geq 3)$ and $x-e^{-x}<x$ $(\forall x\geq 3)$ since $e^{-x}=\frac{1}{e^x}>0$ for all $x\in\mathbb{R}$. Therefore $\frac{1}{x}<\frac{1}{x-e^{-x}}$ (for all $x\geq 3$). By the Direct Compariso Test $\int_3^\infty \frac{dx}{x-e^{-x}}$ diverges since $\int_3^\infty\frac{dx}{x}$ diverges by the p-test $(p=1)$.

OR $\frac{1}{x-e^{-x}} = \frac{e^x}{xe^x-1} > \frac{e^x}{xe^x} = \frac{1}{x}$ $(\forall x\geq 3)$

OR

$$\lim_{x\to+\infty} \frac{\frac{1}{x-e^{-x}}}{\frac{1}{x}} = \lim_{x\to+\infty} \frac{x}{x-e^{-x}} \underset{\uparrow}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to+\infty} \frac{1}{1+e^{-x}} = 1 \Rightarrow$$

L'Hôpital's Rule

By the Limit Comparison Test, $\int_3^\infty \frac{dx}{x-e^{-x}}$ diverges since $\int_3^\infty \frac{dx}{x}$ diverges (p- test, $p=1$).

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c) $x\to 1^+ \Rightarrow \ln x \to 0^+ \Rightarrow \frac{1}{\ln x} \to +\infty$

$x\to 1^+ \Rightarrow x-1 \to 0^+ \Rightarrow \frac{1}{x-1} \to +\infty$

$$\lim_{x\to 1^+}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) \overset{\infty-\infty}{=} \lim_{x\to 1^+} \frac{x-1-\ln x}{x\ln x - \ln x} \underset{\uparrow}{\overset{\frac{0}{0}}{=}} \lim_{x\to 1^+} \frac{1-\frac{1}{x}}{\ln x + 1 - \frac{1}{x}}$$

L'Hôpital's Rule

$$\underset{\uparrow}{\overset{\frac{0}{0}}{=}} \lim_{x\to 1^+} \frac{+\frac{1}{x^2}}{\frac{1}{x}+\frac{1}{x^2}} = \frac{1}{2} \,//$$

L'Hôpital's Rule
