MAT 101E FINAL 30.05.2011

QUESTION 1

Student Number:	e-mail:	Group:	List No.:	Grade
Name:	Surname:		Sign.:	

[12p] a) $\lim_{x\to 0} \frac{e^x - 1 + \int_0^x e^{\tan^{-1} t} dt}{2(1-\cos x)} = ?$

[13p] b) Discuss the convergence of $\int_1^\infty 2e^{-x^2} dx$. Show all work.

a) $\lim_{x\to 0} \frac{e^x - 1 + \int_0^x e^{tg^{-1} t} dt}{2(1-\cos x)} : \left(\frac{0}{0}\right)$

$\lim_{x\to 0} \frac{e^x + e^{tg^{-1} x}}{2\sin x} : \infty$

b) $I = 2\int_1^\infty e^{-x^2} dx$

$1 \le x < \infty$, $x^2 \geq x$, $-x^2 \le -x$

$e^{-x^2} \le e^{-x}$, $g(x) = e^{-x}$
$f(x) = e^{-x^2}$

Let $\int_1^\infty g(x) dx = \int_1^\infty e^{-x} dx$

$\int_1^\infty g(x) dx = \lim_{b\to\infty} \int_1^b e^{-x} dx = \lim_{b\to\infty} \left[-e^{-x} \Big|_1^b \right]$

$= \lim_{b\to\infty} -e^{-b} + \frac{1}{e} = -\frac{1}{e^\infty} + \frac{1}{e} = \frac{1}{e}$

$\int_1^\infty g(x) dx$ converges $\Rightarrow$ $\int_1^\infty f(x) dx$ converges

$\Rightarrow \int_1^\infty 2e^{-x^2} dx$ converges

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QUESTION 2

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Evaluate the following integrals.

[12p] a) $\displaystyle\int_2^{\infty} \frac{(x+3)}{(x-1)(x^2+1)}\,dx$

[13p] b) $\displaystyle\int \frac{\tan^{-1}(2x)}{4x^2}\,dx$

a) $I = \displaystyle\int_2^{\infty} \frac{(x+3)}{(x-1)(x^2+1)}dx$

$$\frac{x+3}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \quad , A=2 \; , B=-2 \; , c=-1$$

$$\frac{x+3}{(x-1)(x^2+1)} = \frac{2}{x-1} - \frac{2x+1}{x^2+1}$$

$$\int_2^{\infty} \frac{(x+3)\,dx}{(x-1)(x^2+1)} = 2\int_2^{\infty} \frac{dx}{x-1} - \int_2^{\infty} \frac{(2x+1)dx}{(x^2+1)}$$

$$= \lim_{b\to\infty} \left(2\int_2^b \frac{dx}{x-1} - \int_2^b \frac{2x\,dx}{x^2+1} - \int_2^b \frac{dx}{x^2+1}\right)$$

$$= \lim_{b\to\infty} \left(2\ln|x-1|\Big|_2^b - \ln|x^2+1|\Big|_2^b - tg^{-1}x\Big|_2^b\right)$$

$$= \lim_{b\to\infty} \left(\ln\frac{(b-1)^2}{(b^2+1)} - 2\overset{0}{\ln 1} + \ln 5 - tg^{-1}b + tg^{-1}2\right)$$

$$= \lim_{b\to\infty} \left[\ln\left(\frac{(b-1)^2}{b^2+1}\right) - tg^{-1}(b)\right] + \ln 5 + tg^{-1}2$$

$$= \ln\left(\lim_{b\to\infty} \frac{b^2-2b+1}{b^2+1}\right) - \underbrace{tg^{-1}(\infty)}_{\frac{\pi}{2}} + \ln 5 + tg^{-1}2$$

$$= \overset{0}{\ln 1} - \frac{\pi}{2} + \ln 5 + tg^{-1}2 = \boxed{\ln 5 + tg^{-1}2 - \frac{\pi}{2}}$$

b) $I = \int \frac{tg^{-1}(2x)}{4x^2} dx$ $\left[tg^{-1}(2x) = u \quad \frac{2dx}{1+4x^2} = du \quad \frac{dx}{4x^2} = dv, \quad v = \frac{-1}{4x} \right]$

$$I = -\frac{tg^{-1}(2x)}{4x} + \frac{2}{4}\int \frac{1}{x} \cdot \frac{dx}{(1+4x^2)}$$

$$\frac{1}{x(1+4x^2)} = \frac{A}{x} + \frac{Bx+C}{1+4x^2}, \quad A=1, \quad B=-4, \quad C=0$$

$$\int \frac{dx}{x(1+4x^2)} = \int \frac{dx}{x} - 4\int \frac{x\,dx}{1+4x^2} \quad \left[\begin{array}{l} 1+4x^2 = m \\ x\,dx = \frac{dm}{8} \end{array} \right]$$

$$= \ln|x| - \cancel{\frac{4}{8}}_{2} \cdot \ln|1+4x^2| + c_1$$

$$\boxed{I = -\frac{tg^{-1}2x}{4x} + \frac{1}{2} \cdot \ln|x| - \frac{1}{4}\ln(1+4x^2) + c}$$

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QUESTION 3

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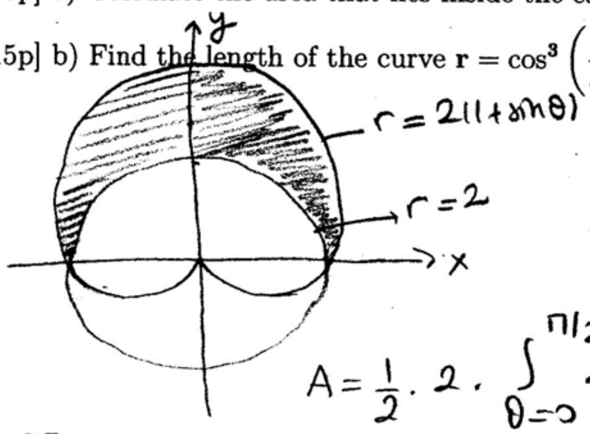
[15p] a) Calculate the area that lies inside the cardioid $\mathbf{r} = \mathbf{2(1+\sin\theta)}$ and outside the circle $\mathbf{r} = \mathbf{2}$.

[15p] b) Find the length of the curve $\mathbf{r} = \cos^3\left(\frac{\theta}{3}\right)$, $0 \le \theta \le \frac{\pi}{4}$.

a)



$$A = \frac{1}{2} \cdot \int_{\theta=\alpha}^{\theta=\beta} r^2 \, d\theta$$

$2(1+\sin\theta) = 2$, $1+\sin\theta = 1$

$\sin\theta = 0$

$\theta_1 = 0$, $\theta_1 = \pi$

$$A = \frac{1}{2} \cdot 2 \cdot \int_{\theta=0}^{\pi/2} 2^2 \cdot (1+\sin\theta)^2 \, d\theta - \frac{\pi \cdot 2^2}{2}$$

or

$$A = \frac{1}{2} \cdot 2 \int_{\theta=0}^{\pi/2} [2^2 (1+\sin\theta)^2 - 2^2] \, d\theta$$

$$A = \int_0^{\pi/2} 4[1+2\sin\theta+\sin^2\theta - 1] \, d\theta = 4\int_0^{\pi/2} 2\sin\theta \, d\theta + 4\int_{\theta=0}^{\pi/2} \sin^2\theta \, d\theta$$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$$A = 8(-\cos\theta)\Big|_0^{\pi/2} + \frac{4}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\Big|_{\theta=0}^{\pi/2}$$

$$A = 8\left(-\overset{0}{\cancel{\cos\frac{\pi}{2}}} + \underbrace{\cos 0}_{1}\right) + 2\left(\frac{\pi}{2} - \overset{0}{\cancel{\frac{\sin\pi}{2}}} - 0 + \frac{\cancel{\sin 0}}{2}\right) = \boxed{8+\pi}$$

b)

$$s = \int_{\theta=\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

$$\frac{dr}{d\theta} = 3\cos^2\frac{\theta}{3} \cdot \frac{1}{3} \cdot \left(-\sin\frac{\theta}{3}\right) = -\cos^2\frac{\theta}{3}\sin\frac{\theta}{3}$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \cos^6\left(\frac{\theta}{3}\right) + \cos^4\frac{\theta}{3} \cdot \sin^2\frac{\theta}{3} = \cos^4\frac{\theta}{3}\left(\cos^2\frac{\theta}{3} + \sin^2\frac{\theta}{3}\right)$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{\cos^4\frac{\theta}{3}} = \left|\cos^2\frac{\theta}{3}\right| = \cos^2\frac{\theta}{3}$$

$$s = \int_{\theta=0}^{\pi/4} \cos^2\frac{\theta}{3} \, d\theta = \frac{1}{2}\int_{\theta=0}^{\pi/4} \left(1 + \cos\frac{2}{3}\theta\right) d\theta$$

$$s = \frac{1}{2} \cdot \left(\theta + \frac{3}{2}\sin\frac{2\theta}{3}\right)\Big|_{\theta=0}^{\pi/4} = \frac{1}{2}\left(\frac{\pi}{4} + \frac{3}{2} \cdot \frac{1}{2}\right) = \boxed{\frac{\pi+3}{8}}$$

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QUESTION 4

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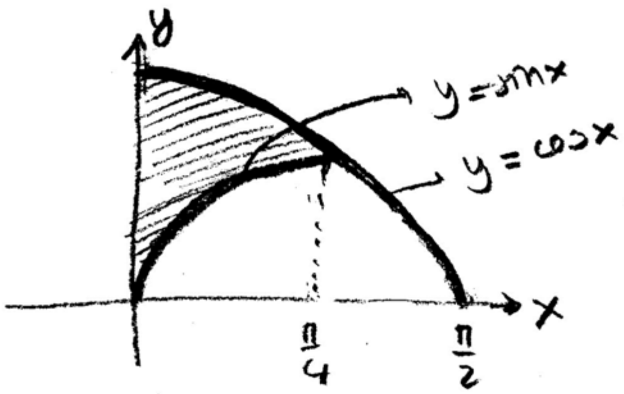
Consider the region in the first quadrant, bounded by the curves $\mathbf{y = \cos x}$, $\mathbf{y = \sin x}$ and the **y**-axes. Find the volume of the solid generated by revolving this region;

[10p] a) about the **y**-axes, by using the shell method.

[10p] b) about the **x**-axes, by using the disk method.



a) $V_y = 2\pi \int_{x_1}^{x_2} xy\,dx$

$V_y = 2\pi \int_0^{\pi/4} x(\cos x - \sin x)\,dx$

$\left[\begin{array}{l} x = u ,\ dx = du \\ (\cos x - \sin x)\,dx = dv \\ (\sin x + \cos x) = v \end{array} \right]$

$V_y = 2\pi \left(x(\cos x + \sin x) \Big|_0^{\pi/4} - \int_0^{\pi/4} (\sin x + \cos x)\,dx \right)$

$V_y = 2\pi \left(\frac{\pi}{4} \cdot 2 \cdot \frac{\sqrt{2}}{2} + (\cos x - \sin x) \Big|_0^{\pi/4} \right)$

$V_y = 2\pi \left(\frac{\sqrt{2}\pi}{4} + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - 1 \right) = 2\pi \left(\frac{\sqrt{2}}{4}\pi - 1 \right)$

b) $V_x = \pi \int_0^{\pi/4} (\cos^2 x - \sin^2 x)\,dx$ $\quad V_x = \pi \int_{x_1}^{x_2} \left[(R(x))^2 - (r(x))^2 \right] dx$

$V_x = \pi \int_0^{\pi/4} \cos 2x\,dx$

$V_x = \pi \frac{\sin 2x}{2} \Big|_0^{\pi/4} = \boxed{\frac{\pi}{2}}$