MATHEMATICS 101E MIDTERM 6 April 2013

QUESTION 1 The blanks below will be filled by students. (Except the score)

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For the solution of this question please use only the front face and if necessary the back face of this page.

(*7 pts*) a) Find $\tan^{-1}(\tan\frac{3\pi}{4}) + \sin^{-1}(\sin\frac{\pi}{4}) + \sin(\cos^{-1}\frac{3}{5}) = ?$
(**Hint:** Consider the domains of the inverse trigonometric functions.)

(*18 pts*) b) Evaluate the following limits. (Do not use the L'Hopital's Rule)

(*9 pts*) (i) $\lim_{x\to 4} \ln\left(\frac{4x - x^2}{2-\sqrt{x}}\right) = ?$

(*9 pts*) (ii) $\lim_{x\to +\infty} \frac{2x+\sin x}{x+1} = ?$

a) $\tan^{-1}(\tan\frac{3\pi}{4}) = \tan^{-1}(-1) = -\frac{\pi}{4}$

$\sin^{-1}(\sin\frac{\pi}{4}) = \frac{\pi}{4}$

$\sin(\cos^{-1}\frac{3}{5}) = \frac{4}{5}$

$\tan^{-1}(\tan\frac{3\pi}{4}) + \sin^{-1}(\sin\frac{\pi}{4}) + \sin(\cos^{-1}\frac{3}{5}) = -\frac{\pi}{4} + \frac{\pi}{4} + \frac{4}{5} = \underline{\underline{\frac{4}{5}}}$

b) (i) $\lim_{x\to 4} \ln\left(\frac{4x-x^2}{2-\sqrt{x}}\right) = \ln\left[\lim_{x\to 4}\left(\frac{4x-x^2}{2-\sqrt{x}}\right)\right]$ since $y=\ln x$ is continuous

$= \ln\left[\lim_{x\to 4}\left(\frac{4x-x^2}{2-\sqrt{x}}\right)\left(\frac{2+\sqrt{x}}{2+\sqrt{x}}\right)\right] = \ln\left[\lim_{x\to 4}\frac{x\cancel{(4-x)}(2+\sqrt{x})}{\cancel{(4-x)}}\right]$

$= \ln[4\cdot(2+\sqrt{4})] = \underline{\underline{\ln 16}}$

(ii) $\lim_{x\to +\infty} \frac{2x+\sin x}{x+1} = \lim_{x\to +\infty} \frac{x(2+\overbrace{\frac{\sin x}{x}}^{x\to+\infty\ 0})}{x\cdot(1+\underbrace{\frac{1}{x}}_{0})} = \underline{\underline{2}}$

QUESTION 2 The blanks below will be filled by students. (Except the score)

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(*12 pts*) a) Is the function
$$f(x) = \begin{cases} \dfrac{\sin|x|}{x} & \text{if } -\dfrac{\pi}{2} < x < 0 \\ 1 & \text{if } x = 0 \\ x+1 & \text{if } 0 < x < 1 \\ 3 & \text{if } x = 1 \\ \sec(\dfrac{\pi x}{3}) & \text{if } 1 < x < \dfrac{4}{3} \end{cases}$$
continuous on the open interval $(-\frac{\pi}{2}, \frac{4}{3})$?

If not, find the points of discontinuity of the function **f** and determine their types by evaluating the one-sided limits.

(*13 pts*) b) (i) State the Intermediate Value Theorem.(*3 pts*)

(ii) Using Intermediate Value Theorem, show that the equation $e^x + 2x = e$ has at least one solution (*10 pts*)

a) Since $y = \frac{\sin|x|}{x}$ is cts on $(-\pi/2, 0)$, $y = x+1$ is cts on $(0,1)$, $y = \sec(\frac{\pi x}{3})$ is cts on $(1, \frac{4}{3})$; we will consider the points $x=0,1$

At $x=0$: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin|x|}{x} = \lim_{x\to 0^-} \frac{\sin(-x)}{x} = \lim_{x\to 0^-} \frac{-\sin x}{x} = -1$

& $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x+1) = 1 = f(0) \Rightarrow f$ is not cts at $x=0$, since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.

TYPE: "Jump" discontinuity.

At $x=1$: $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x+1) = 2$ & $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \sec(\frac{\pi x}{3}) = 2$

$\Rightarrow \lim_{x\to 1} f(x) = 2$. Since $f(1) = 3 \neq \lim_{x\to 1} f(x)$, f is not cts at $x=1$

TYPE: "Removable" discontinuity.

b) (i) The Intermediate Value Theorem: A function $y=f(x)$ that is continuous on a closed interval $[a,b]$ takes on every value between $f(a)$ and $f(b)$.

OR

Let f be a continuous function on the closed interval $[a,b]$. If y_0 is any value between $f(a)$ and $f(b)$, then $y_0=f(c)$ for some $c\in[a,b]$.

(ii) Let $f(x)=e^x+2x-e$. Then f is continuous everywhere.

$$\left.\begin{array}{l} f(0)=1-e<0 \\ f(1)=e+2-e=2>0 \end{array}\right\}$$ By (a corrollary to) the Intermediate Value Theorem f has at least one zero in $(0,1)$.

Therefore, $e^x+2x-e=0$ has at least one solution.

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QUESTION 3 The blanks below will be filled by students. (Except the score)

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(10 pts) **a)** Let $\mathbf{f}$ be a function satisfying the equation $\mathbf{f(x+h) = f(x) + f(h) + x^2h + xh^2}$ for all $\mathbf{x, h} \in \mathbb{R}$. Suppose that $\lim\limits_{\mathbf{h\to 0}} \frac{\mathbf{f(h)}}{\mathbf{h}} = \mathbf{1}$.

(i) Show that $\mathbf{f(0) = 0}$.

(ii) By using the limit definition of the derivative find $\mathbf{f'(x)}$.

(5 pts) **b)** Find $\frac{\mathbf{dy}}{\mathbf{dx}}$ where $\mathbf{y = (\sec x + \tan x)^2 + (x^2+1)^{5/2}}$.

(10 pts) **c)** Find the equation of the tangent line to the parameterized curve $\mathbf{x(t) = \sin t}$, $\mathbf{y(t) = \sqrt{t+1}}$ at $\mathbf{t = 0}$.

a)(i) $f(\underset{x}{0}+\underset{h}{0}) = f(0) + f(0) + 0^2.0 + 0.0^2 = 0 \Rightarrow f(0) = 2f(0)$
$\Rightarrow f(0) = 0$

(ii) $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{f(x)+f(h)+x^2h+xh^2-f(x)}{h}$

$= \lim\limits_{h\to 0} \dfrac{f(h)+h(x^2+xh)}{h} = \lim\limits_{h\to 0}\left[\dfrac{f(h)}{h} + (x^2+xh)\right] = 1+x^2$

since $\lim\limits_{h\to 0} \dfrac{f(h)}{h} = 1$.

b) $y = (\sec x + \tan x)^2 + (x^2+1)^{5/2}$

$\dfrac{dy}{dx} = 2(\sec x + \tan x) \cdot (\sec x + \tan x)' + \frac{5}{2}(x^2+1)^{3/2} \cdot (x^2+1)'$
$= 2(\sec x + \tan x) \cdot (\sec x \tan x + \sec^2 x) + \frac{5}{2}(x^2+1)^{3/2} \cdot 2x$

c) $x(t) = \sin t \Rightarrow \dfrac{dx}{dt} = \cos t$
$y(t) = \sqrt{t+1} \Rightarrow \dfrac{dy}{dt} = \dfrac{1}{2\sqrt{t+1}}$

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\frac{1}{2\sqrt{t+1}}}{\cos t} = \dfrac{1}{2\sqrt{t+1}.\cos t}$

$m_{tangent} = \left.\dfrac{dy}{dx}\right|_{t=0} = \dfrac{1}{2}$

$t=0 \Rightarrow x=0$, $y=1$

$P(0,1)$, $m_{tangent} = 1/2 \Rightarrow y-1 = \frac{x}{2} \Rightarrow \boxed{y = \frac{x}{2}+1}$ is the tangent line at P

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QUESTION 4　　　The blanks below will be filled by students. (Except the score)

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(25 pts) Let $\mathbf{f(x)} = \dfrac{\mathbf{x^2+1}}{\mathbf{(x-1)^2}}$.

(i) Find the domain and, if any, the intercept(s) of $\mathbf{f(x)}$. $D_f: (-\infty, +1) \cup (+1, +\infty)$

$x=0 \Rightarrow y=1 \Rightarrow (0,1)$ is the y-intercept of f.

Since $x^2+1>0$ for any $x \in D_f$, there is no x-intercept for f.

(ii) If any, find the horizontal asymptote(s) of $\mathbf{f(x)}$.

$\lim\limits_{x\to\pm\infty} \dfrac{x^2+1}{(x-1)^2} = 1 \Rightarrow y=1$ is the horizontal asymptote of f.

(iii) If any, find the vertical asymptote(s) of $\mathbf{f(x)}$.

$\lim\limits_{x\to 1^-} \dfrac{x^2+1}{(x-1)^2} = +\infty$, $\lim\limits_{x\to 1^-} \dfrac{x^2+1}{(x-1)^2} = +\infty \Rightarrow x=1$ is the vertical asymptote of f.

(iv) If any, find the oblique asymptote(s) of $\mathbf{f(x)}$.

There is <u>no</u> oblique asymptote for f.

(v) Find the intervals on which the function is increasing and decreasing, and identify the function's local extreme value(s), if any, saying where they are taken on.

$$f'(x) = \frac{2x(x-1)^2 - (x^2+1)2(x-1)}{(x-1)^4} = \frac{2\cancel{(x-1)}[x(x-1)-(x^2+1)]}{(x-1)^{\cancel{4}3}} = \frac{2(x^2-x-x^2-1)}{(x-1)^3}$$

$\Rightarrow f'(x) = -\dfrac{2(x+1)}{(x-1)^3} = 0 \Rightarrow x=-1$ critical point, $x \neq 1$

		-1		1	
$-(x+1)$	$+$	0	$-$	‖	$-$
$(x-1)^3$	$-$		$-$	0	$+$
f'	$-$		$+$	‖	$-$

- f is increasing on $(-1,1)$
- f is decreasing on $(-\infty,-1)\cup(1,\infty)$
- f has a local minimum at $x=-1$, $f(-1)=1/2$

(vi) Identify the concavity and, if any, find the point(s) of inflection.

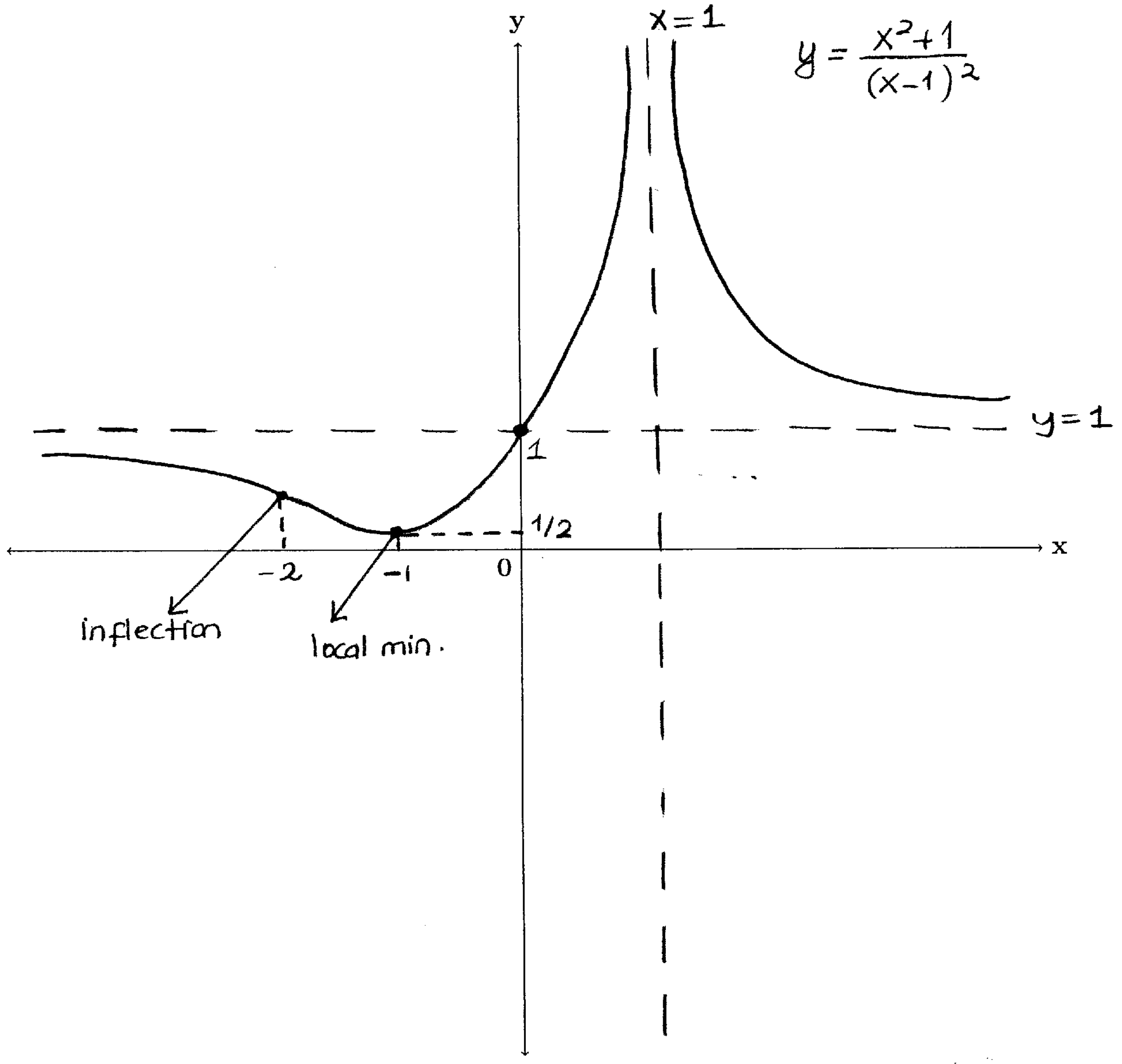
$$f''(x) = \frac{-2(x-1)^3 + 2(x+1)3(x-1)^2}{(x-1)^6} = \frac{2\cancel{(x-1)^2}[3(x+1)-(x-1)]}{(x-1)^{\cancel{6}4}} = \frac{4x+8}{(x-1)^4} = 0 \Rightarrow x=-2,\ x\neq 1$$

		-2		1	
f''	$-$	0	$+$	‖	$+$

- f is concave down on $(-\infty,-2)$.
- f is concave up on $(-2,\infty)$ and $x=-2$ is an inflection point.

(vii) By using all obtained above, fill in the following table and **graph the curve of $\mathbf{y=f(x)}$ on the back side of the paper.**

	$-\infty$	-2		-1		1	$+\infty$
f'	$-$		$-$	0	$+$	‖	$-$
f''	$-$	0	$+$		$+$	‖	$+$
	$y=1$				$+\infty$	‖ $+\infty$	$y=1$


y
x = 1
$y = \frac{x^2+1}{(x-1)^2}$
y = 1
1
1/2
x
-2
-1
0
inflection
local min.