MATHEMATICS 101E | MIDTERM | 16 July 2013

QUESTION 1 The blanks below will be filled by students. (Except the score)

Surname:	Name:	Group Number:	List Number:	Score
Signature:	Electronic Post(e-mail) address:	Student Number:		

For the solution of this question please use only the front face and if necessary the back face of this page.

[17 pts] a) Evaluate the following limits (**Do not use L'Hopital's Rule**).

[6 pts] i) $\lim_{x\to 0}\left(\frac{\sin 4x - 2\sin 2x}{x^3}\right) = ?$

[6 pts] ii) $\lim_{x\to\infty}\left(2x - \sqrt{4x^2-3x}\right) = ?$

[5 pts] iii) $\lim_{x\to\frac{\pi}{4}}\left(\frac{\sec^2 x - 2}{x - \frac{\pi}{4}}\right) = ?$

[8 pts] b) Let $F(x) = (f\circ g)\ x)$. Find $f'(3)$ by using the following information:

- The equation of the tangent line to the graph of the function $g(x)$ at the point $(5,3)$ is $y = 4x - 17$.
- $\lim_{h\to 0}\frac{F(5+h)-F(5)}{h} = 2$

(**Hint:** Use the Chain Rule.)

a.i) $\lim_{x\to 0}\frac{\sin 4x - 2\sin 2x}{x^3} = \lim_{x\to 0}\frac{2\sin 2x\cos 2x - 2\sin 2x}{x^3}$

$= \lim_{x\to 0}\frac{2\sin 2x\,(\cos 2x - 1)}{x^3} = \lim_{x\to 0}\frac{2\sin 2x\cdot(-2\sin^2 x)}{x^3}$

$= -4\lim_{x\to 0}\left[\frac{\sin 2x}{x}\cdot\left(\frac{\sin x}{x}\right)^2\right] = -4.2.1^2 = -8$

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a.ii) $\lim_{x\to\infty}\left(2x - \sqrt{4x^2-3x}\right) = \lim_{x\to\infty}\left(2x - \sqrt{4x^2-3x}\right)\frac{\left(2x+\sqrt{4x^2-3x}\right)}{2x+\sqrt{4x^2-3x}}$

$= \lim_{x\to\infty}\frac{4x^2 - (4x^2-3x)}{2x+\sqrt{4x^2-3x}} = \lim_{x\to\infty}\frac{3x}{2x+|2x|\sqrt{1-\frac{3x}{4x^2}}}$

$= \lim_{x\to\infty}\frac{3\cancel{x}}{\cancel{x}\left[2+2\sqrt{1-\underbrace{\frac{3}{4x}}_{0}}\right]} = \frac{3}{2+2.1} = \frac{3}{4}$

a.iii) $\lim_{x\to\frac{\pi}{4}} \dfrac{\sec^2 x - 2}{x-\frac{\pi}{4}} = \lim_{x\to\frac{\pi}{4}} \dfrac{\frac{1}{\cos^2 x} - 2}{x-\frac{\pi}{4}} = \lim_{x\to\frac{\pi}{4}} \dfrac{1-2\cos^2 x}{\cos^2 x\,(x-\frac{\pi}{4})}$

$= \lim_{x\to\frac{\pi}{4}} \dfrac{-\cos 2x}{\cos^2 x.(x-\frac{\pi}{4})}$ $\qquad x-\frac{\pi}{4}=t$

$= \lim_{t\to 0} \dfrac{-\cos 2(\frac{\pi}{4}+t)}{\cos^2(t+\frac{\pi}{4}).\ t} = \lim_{t\to 0} \dfrac{-\cos(\frac{\pi}{2}+2t)}{t\ \cos^2(t+\frac{\pi}{4})}$

$= \lim_{t\to 0} \dfrac{-(-\sin 2t)}{t\cos^2(t+\frac{\pi}{4})} = \lim_{t\to 0} \left\{\dfrac{\sin 2t}{t} \cdot \dfrac{1}{\cos^2(t+\frac{\pi}{4})}\right\}$

$= 2.\dfrac{1}{(1/\sqrt{2})^2} = 4$

OR

$\lim_{x\to\frac{\pi}{4}} \dfrac{\sec^2 x - 2}{x-\frac{\pi}{4}} = \lim_{x\to\frac{\pi}{4}} \dfrac{\sec^2 x - (\sec\frac{\pi}{4})^2}{x-\frac{\pi}{4}}$

$= \dfrac{d}{dx}(\sec^2 x)\Big|_{x=\frac{\pi}{4}} = (2.\sec x.\tan x.\sec x)\Big|_{x=\frac{\pi}{4}}$

$= 2.(\sec\frac{\pi}{4})^2.\tan\frac{\pi}{4} = 2.(\sqrt{2})^2.1$

$= 4$ //

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b) $F(x) = f\circ g\,(x) = f(g(x)) \Rightarrow F'(x) = f'(g(x)).g'(x)$ (*)

- $g'(5) = 4$, $g(5) = 3$
- $\lim_{h\to 0} \dfrac{F(5+h)-F(5)}{h} = F'(5) = 2$

Plug $x=5$ in (*) : $F'(5) = f'(g(5)).\,g'(5)$

$2 = f'(3).\,4$

$f'(3) = \dfrac{1}{2}$ //

QUESTION 2 The blanks below will be filled by students. (Except the score)

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[15 pts] **a)** Is the function

$$f(x)=\begin{cases} \dfrac{|\sin x|}{x} & ; \; x<0 \\ \tan^{-1}\dfrac{x+2}{x-2} & ; \; 0\le x<2 \\ \dfrac{\pi}{2} & ; \; x=2 \\ \sin^{-1}\dfrac{-2}{x} & ; \; 2<x\le 4 \\ \dfrac{1}{4-x} & ; \; 4<x \end{cases}$$

continuous everywhere?

If not, find the points of discontinuity of the function f and identify the types of discontinuity by evaluating the one-sided limits.

[10 pts] **b)** Suppose that $y=2x+1$ is tangent to $y=f(x)$ at $(1,3)$ and $y=x+4$ is tangent to $y=g(x)$ at $(2,6)$. Find the slope of the tangent line to the curve given by the equation $f(x)g(y)=18$ at the point $(1,2)$ by using **implicit differentiation**.

a) $\dfrac{|\sin x|}{x}$ is continuous for $x<0$

$\tan^{-1}\dfrac{x+2}{x-2}$ is continuous for $0<x<2$

$\sin^{-1}\left(\dfrac{-2}{x}\right)$ is continuous for $2<x<4$

$\dfrac{1}{4-x}$ is continuous for $4<x$

We need to check continuity of $f(x)$ at $x=\{0,2,4\}$

$\boxed{x=0}$: $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}\dfrac{|\sin x|}{x}=\lim\limits_{x\to 0^-}\dfrac{-\sin x}{x}=-1$

$\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+}\tan^{-1}\left(\dfrac{x+2}{x-2}\right)=\tan^{-1}(-1)=-\dfrac{\pi}{4}$

$\lim\limits_{x\to 0} f(x)$ does not exist $\Rightarrow$ f is not cont. at $x=0$.

JUMP DISCONTINUITY

$\boxed{x=2}$ $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \tan^{-1}\left(\frac{x+2}{x-2}\right) = \tan^{-1}(-\infty) = -\frac{\pi}{2}$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \sin^{-1}\left(\frac{-2}{x}\right) = \sin^{-1}(-1) = -\frac{\pi}{2}$

$\therefore \lim_{x\to 2} f(x) = -\frac{\pi}{2} \neq f(2) = \frac{\pi}{2}$ $\Rightarrow$ $f(x)$ is not cont. at $x=2$.

REMOVABLE DISCONTINUITY

$\boxed{x=4}$ $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \sin^{-1}\left(\frac{-2}{x}\right) = \sin^{-1}\left(\frac{-2}{4}\right) = -\frac{\pi}{6}$

$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \frac{1}{4-x} = -\infty$

$\therefore \lim_{x\to 4} f(x)$ does not exist $\Rightarrow$ $f(x)$ is not cont. at $x=4$.

INFINITE DISCONTINUITY

(b) $y=2x+1$ is tangent to $y=f(x)$ at $(1,3)$: $f(1)=3$, $f'(1)=2$

$y=x+4$ is tangent to $y=g(x)$ at $(2,6)$: $g(2)=6$, $g'(2)=1$

$f(x)\,g(y)=18$. By implicit differentiation we find

$f'(x)\cdot g(y) + f(x)\cdot y'\cdot g'(y) = 0$ For $x=1$, $y=2$ we get

$f'(1)\cdot g(2) + f(1)\cdot y'\big|_{(1,2)}\cdot g'(2) = 0$

$2\cdot 6 + 3\cdot y'\big|_{(1,2)}\cdot 1 = 0 \Rightarrow y'\big|_{(1,2)} = -4$ = slope of the tangent at $(1,2)$

QUESTION 3 The blanks below will be filled by students. (Except the score)

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[13 pts] a) Find an equation for the line that is tangent to the curve **C** at the point corresponding to the given value of t. **Find also** $\frac{d^2y}{dx^2}$ **at this point:**

C: $x = \tan t$; $y = \sec t$; $t = \frac{\pi}{3}$.

[12 pts] b) For what values of a and b, does the function

$$f(x) = \begin{cases} ax + 4\pi & ; \quad -\pi \le x \le 0 \\ b\cos 2x \; + \; 2x & ; \quad 0 < x \le \pi \end{cases}$$

satisfy the hypotheses of the Mean Value Theorem on $[-\pi, \pi]$?

CAUTION: If one sided derivatives are necessary, use the limit definition of the derivative.

(a) $t = \frac{\pi}{3}$: $x = \tan\frac{\pi}{3} = \sqrt{3}$, $y = \sec\frac{\pi}{3} = 2$

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\tan t \sec t}{\sec^2 t} = \sin t, \quad \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{3}} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$y = 2 + \frac{\sqrt{3}}{2}(x - \sqrt{3}) = \frac{\sqrt{3}}{2}x + \frac{1}{2}$ the equation of the tangent line.

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{\cos t}{\sec^2 t} = (\cos t)^3 \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{3}} = \left(\frac{1}{2}\right)^3 = \frac{1}{8} \,//$$

(b) $f(x)$ must be continuous on $[-\pi, \pi]$ and differentiable on $(-\pi, \pi)$.

Continuity at $x=0$: $\lim_{x\to 0^+} f = \lim_{x\to 0^-} f = f(0)$

$$\lim_{x\to 0^+}(b\cos 2x + 2x) = \lim_{x\to 0^-}(ax + 4\pi) = a.0 + 4\pi \Rightarrow \quad b = 4\pi \,//$$

Differentiability at $x=0$: $f'_+(0) = f'_-(0)$ must hold.

$$f'_+(0) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{4\pi \cos 2h + 2h - 4\pi}{h}$$

$$= \lim_{h\to 0^+} \left\{ \frac{4\pi(\cos 2h - 1)}{h} + 2 \right\} = \lim_{h\to 0^+} \left\{ 4\pi . \frac{-2\sin^2 h}{h} + 2 \right\}$$

$$= \lim_{h\to 0^+} \left\{ -8\pi . \frac{\sin h}{h} . \sin h + 2 \right\} = -8\pi . 1 . 0 + 2 = 2$$

$$f'_-(0) = \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{ah + 4\pi - 4\pi}{h} = a$$

$$\Rightarrow \quad a = 2 \quad //$$

QUESTION 4 The blanks below will be filled by students. (Except the score)

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[25 pts] For the function $f(x) = \dfrac{x^3-3}{(x-1)^3}$;

i) Find the domain and, if any, the intercept(s) of the function $f(x)$.

Domain $= \mathbb{R}\setminus\{1\} = (-\infty,1)\cup(1,\infty)$ $\qquad (\sqrt[3]{3}, 0)$: x-intercept

$x=0 \Rightarrow y=3$, $\quad y=0 \Rightarrow x=\sqrt[3]{3}$ $\qquad (0,3)$: y-intercept

ii) Find all asymptote(s) of the function $f(x)$, if any.

- $\lim\limits_{x\to\mp\infty} \dfrac{x^3-3}{(x-1)^3} = 1 \Rightarrow y=1$ is a horizontal asymptote.
- No oblique asymptote
- $\lim\limits_{x\to 1^+} \dfrac{x^3-3}{(x-1)^3} = -\infty$, $\lim\limits_{x\to 1^-} \dfrac{x^3-3}{(x-1)^3} = +\infty \Rightarrow x=1$ is a vertical asymptote.

iii) Find the intervals on which the function is increasing and decreasing, and identify the function's local extreme value(s), if any, saying where they are taken on.

$$f'(x) = \frac{3x^2.(x-1)^3-(x^3-3).3.(x-1)^2}{(x-1)^6} = \frac{9-3x^2}{(x-1)^4} = \frac{-3(x-\sqrt{3})(x+\sqrt{3})}{(x-1)^4}$$

		$-\sqrt{3}$		1		$\sqrt{3}$	
f'	$-$	0	$+$		$+$	0	$-$
	↘		↗		↗		↘

f is increasing on $(-\sqrt{3},1)$ and $(1,\sqrt{3})$

f is decreasing on $(-\infty,-\sqrt{3})$ and $(\sqrt{3},\infty)$

$\left(-\sqrt{3}, \dfrac{3}{(\sqrt{3}+1)^2}\right)$ local minimum point

$\left(\sqrt{3}, \dfrac{3}{(\sqrt{3}-1)^2}\right)$ local maximum point

iv) Identify the concavity and, if any, find the point(s) of inflection.

$$f''(x) = \frac{-6x.(x-1)^4-(9-3x^2).4.(x-1)^3}{(x-1)^8} = \frac{6.(x-2)(x+3)}{(x-1)^5}$$

		-3		1		2	
f''	$-$	0	$+$		$-$	0	$+$
	conc. down $\cap$		conc. up $\cup$		conc. down $\cap$		conc. up $\cup$

f is concave up on $(-3,1)$ and $(2,\infty)$

f is concave down on $(-\infty,-3)$ and $(1,2)$

$\left(-3, \dfrac{15}{32}\right)$ and $(2,5)$ are inflection points

v) By using all obtained above, complete the table given on the back side of the paper and then draw the graph of the curve of the function $y=f(x)$ on the back side.

→

x	$-\infty$		-3		$-\sqrt{3}$		1		$\sqrt{3}$		2		$+\infty$
f'		$-$		$-$	0	$+$	0	$+$	0	$-$		$-$	
f''		$-$	0	$+$		$+$	0	$-$		$-$	0	$+$	
	$y=1$						$+\infty$ ‖ $-\infty$						$y=1$