MATHEMATICS 101E — MIDTERM — JULY 14, 2010

QUESTION 1:

Surname:	Name:	Group No.:	List No.:	Score
Sign:	e-mail:	Student Number:		

[12pt.] **a)** Find all values of the constants a and b for which the function $f(x)=\begin{cases} x^4+x+1 & x\geq 1; \\ ax+b & x<1 \end{cases}$ is differentiable.

[13pt.] **b)** $\lim\limits_{x\to 0}\dfrac{3x\sin x}{2\cos x-2}=?$ (Do not use L' Hospital's rule)

a) $f'(x)=\begin{cases} 4x^3+1 &, x\geqslant 1 \\ a &, x<1 \end{cases}$

For differentiability :

* $f'(1)=f'_-(1) \Rightarrow f'(1)=4\cdot 1^3+1=5=a \Rightarrow a=5$
* (Continuity) $f(1)=\lim\limits_{x\to 1^-} f(x) \Rightarrow f(1)=1^4+1+1=3=\lim\limits_{x\to 1} 5x+b=5+b \Rightarrow b=-2$

b) $\lim\limits_{x\to 0}\dfrac{3x\sin x}{2\cos x-2} \overset{\frac{0}{0}}{=} \dfrac{3}{-2}\cdot\lim\limits_{x\to 0}\dfrac{x\sin x}{(1-\cos x)}\cdot\dfrac{(1+\cos x)}{(1+\cos x)} = -1.5\lim\limits_{x\to 0}\dfrac{x\sin x\,(1+\cos x)}{1-\cos^2 x}$

$=-1.5\lim\limits_{x\to 0}\dfrac{x\cdot\sin x}{\sin^2 x}(1+\cos x) = -1.5\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{-1}(1+\cos x)$

$=-1.5\cdot 1^{-1}\cdot(1+1) = -3$

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QUESTION 2:

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[12pt.] **a)** For the curve given by the equation $x^2y+y^3+x^2=8$ find all points on the curve where its tangent line is horizontal.

[13pt.] **b)** Does the function, $f(x)=\sqrt{-2x^2+11x-12}$ satisfy hypotheses of Rolle's theorem on the interval $\left[\frac{3}{2},4\right]$? If so, find an admissible value for $c\in\left[\frac{3}{2},4\right]$.

a) Deriving $x^2y+y^3+x^2=8$ implicitly with respect to x gives:

$$2xy+x^2\cdot y'+3y^2y'+2x=0$$

$$y'(x^2+3y^2)+2x(y+1)=0$$

$$y'=\frac{-2x(y+1)}{x^2+3y^2} \qquad (x,y)\neq(0,0)$$

Horizontal tangent line $\Rightarrow y'=0 \iff x=0 \vee y=-1$

$x=0$: $0^2y+y^3+0^2=y^3=8 \Rightarrow y=2 \Rightarrow \underline{(0,2)}$

$y=-1$: $x^2(-1)+(-1)^3+x^2=-1=8$ ↯ There is no point on the curve with $y=-1$

b) $f(x)=\sqrt{-2x^2+11x-12}=\sqrt{(-2x+3)(x-4)}$

$(-2x+3)(x-4)=0 \iff x=1.5 \vee x=4$

$(-2x+3)(x-4)\geqslant 0$ if $1.5\leqslant x\leqslant 4$, so $f(x)$ is defined and continuous on $[1.5;4]$ (I)

$f'(x)=\dfrac{-4x+11}{2\sqrt{-2x^2+11x-12}}$) $f'(x)$ exists if $x\neq 1.5 \wedge x\neq 4$, so $f'(x)$ exists on $(1.5;4)$ (II)

$f(1.5)=0 \quad f(4)=0 \Rightarrow f(1.5)=f(4)$ (III)

$f(x)$ satisfies the hypotheses (I, II, III) of Rolle's theorem on the given interval.

$\exists c\in(1.5;4) \mid f'(c)=0$.

$$0=f'(c)=\frac{-4c+11}{2\sqrt{-2c^2+11c-12}} \Rightarrow c=\frac{11}{4}=\underline{2.75}$$

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QUESTION 3:

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[12pt.] **a)** Use the definition of the derivative to compute the derivative of the function $f(x)=x+\frac{1}{x}$ for all points $x>0$.

[13pt.] **b)** Find the equations of the tangent and normal lines to the curve given by the parameterized equations $x=2\sec t$, $y=\sqrt{3}\tan t$ at the point $t=\pi/6$.

a) $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{x+h+\frac{1}{x+h}-x-\frac{1}{x}}{h}=\lim_{h\to 0}\frac{h+\frac{-h}{x(x+h)}}{h}$$

$$=\lim_{h\to 0}\left(1-\frac{1}{x(x+h)}\right)=1-\frac{1}{x^2}$$

b) $$x=2\sec t=\frac{2}{\cos t} \qquad \frac{dx}{dt}=\frac{0\cdot\cos t-2(-\sin t)}{\cos^2 t}=\frac{2\sin t}{\cos^2 t}$$

$$y=\sqrt{3}\tan t \qquad \frac{dy}{dt}=\sqrt{3}\sec^2 t=\frac{\sqrt{3}}{\cos^2 t}$$

$$y'=\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\frac{\sqrt{3}}{\cos^2 t}}{\frac{2\sin t}{\cos^2 t}}=\frac{\sqrt{3}}{2\sin t} \qquad y'\Big|_{t=\frac{\pi}{6}}=\frac{\sqrt{3}}{2\sin\frac{\pi}{6}}=\sqrt{3}$$

$$x\Big|_{t=\frac{\pi}{6}}=\frac{2}{\cos\frac{\pi}{6}}=\frac{2}{\frac{\sqrt{3}}{2}}=\frac{4\sqrt{3}}{3} \qquad y\Big|_{t=\frac{\pi}{6}}=\sqrt{3}\tan\frac{\pi}{6}=\sqrt{3}\cdot\frac{\sqrt{3}}{3}=1$$

Tangent line: $m_t=y'\Big|_{t=\frac{\pi}{6}}=\sqrt{3}$ $\qquad P\left(\frac{4\sqrt{3}}{3},1\right)$

$$y=\sqrt{3}x-3$$

Normal line: $m_n=\frac{-1}{m_t}=\frac{-1}{\sqrt{3}}=-\frac{\sqrt{3}}{3}$ $\qquad P\left(\frac{4\sqrt{3}}{3},1\right)$

$$y=-\frac{\sqrt{3}}{3}x+\frac{7}{3}$$

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QUESTION 4:

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[25pt.] Sketch the graph of the function $f(x)=\dfrac{x^2-3}{|x-2|}$ by determining the domain of the function, intercepts, symmetries, asymptotes, intervals on which the function is decreasing or increasing, extremums and inflection points (if they exist) and concavity.

$$f(x)=\begin{cases}\dfrac{x^2-3}{x-2}, & x>2\\[2mm] \dfrac{-x^2+3}{x-2}, & x<2\end{cases}\qquad D: \mathbb{R}\setminus\{2\}$$

$f(x)=0 \iff x^2-3=0 \iff x=\pm\sqrt{3}$ x-intercepts: $(\sqrt{3},0)$, $(-\sqrt{3},0)$

$f(0)=\dfrac{0^2-3}{|0-2|}=-1.5$ y-intercept: $(0,-1.5)$

$f(-x)=\dfrac{(-x)^2-3}{|-x-2|}=\dfrac{x^2-3}{|x+2|}$ $f(-x)\neq f(x)$, $f(-x)\neq -f(x)$ no symmetry

$\lim\limits_{x\to\pm\infty} f(x)=\lim\limits_{x\to\pm\infty}\dfrac{x^2-3}{|x-2|}=\infty$ no horizontal asymptote

$\lim\limits_{x\to 2} f(x)=\lim\limits_{x\to 2}\dfrac{x^2-3}{|x-2|}=\infty$ vertical asymptote: $x=2$

$\dfrac{x^2-3}{x-2}=\dfrac{x^2-4+1}{x-2}=\underline{x+2}+\dfrac{1}{x-2}$ oblique asymptotes: $y_1=x+2$, $y_2=-x-2$

$\dfrac{-x^2+3}{x-2}=\underline{-x-2}-\dfrac{1}{x-2}$

$$f'(x)=\begin{cases}1-\dfrac{1}{(x-2)^2}, & x>2\\[2mm] -1+\dfrac{1}{(x-2)^2}, & x<2\end{cases}$$

$f'(x)=0 \iff x=1 \lor x=3$

$f'(x)$ undefined if $x=2$

Critical x-values: $\{1;\ 2;\ 3\}$

		1		2		3	
f'	$-$	min	$+$	$\times$	$-$	min	$+$
f	$\searrow$		$\nearrow$		$\searrow$		$\nearrow$

$f\ \searrow$ on $(-\infty;1)\cup(2;3)$

$f\ \nearrow$ on $(1;2)\cup(3;\infty)$

$f(1) = \dfrac{1^2-3}{|1-2|} = -2$ $\qquad$ $f(3) = \dfrac{3^2-3}{|3-2|} = 6$ $\qquad$ Minima: $(1,-2)$, $(3,6)$

no maximum

As $\lim\limits_{x\to\pm\infty} f(x) = \infty$, $(1,-2)$ is abs. min. and there is no abs. max.

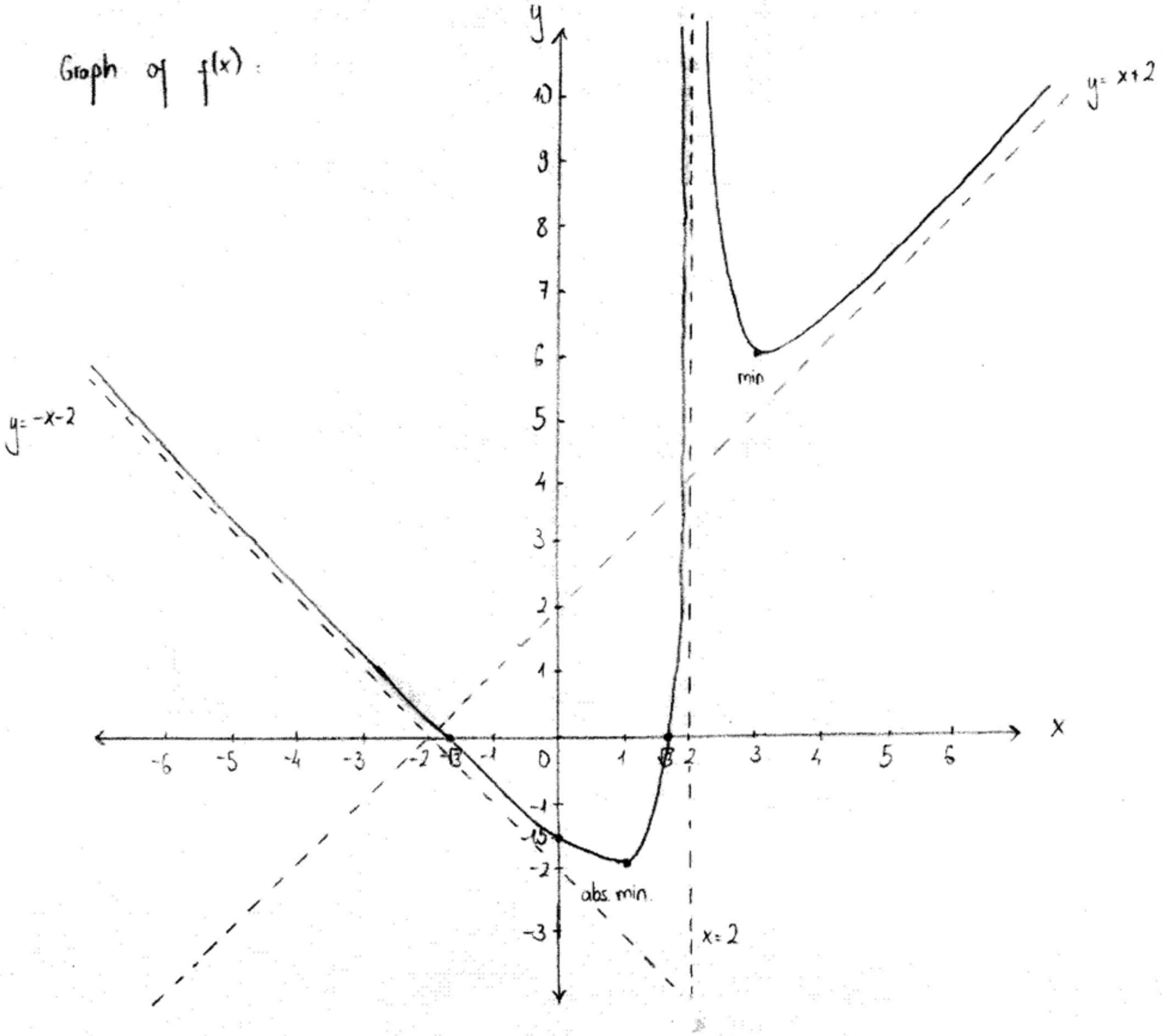
$$f''(x) = \begin{cases} \dfrac{2}{(x-2)^3}, & x>2 \\ \dfrac{-2}{(x-2)^3}, & x<2 \end{cases}$$

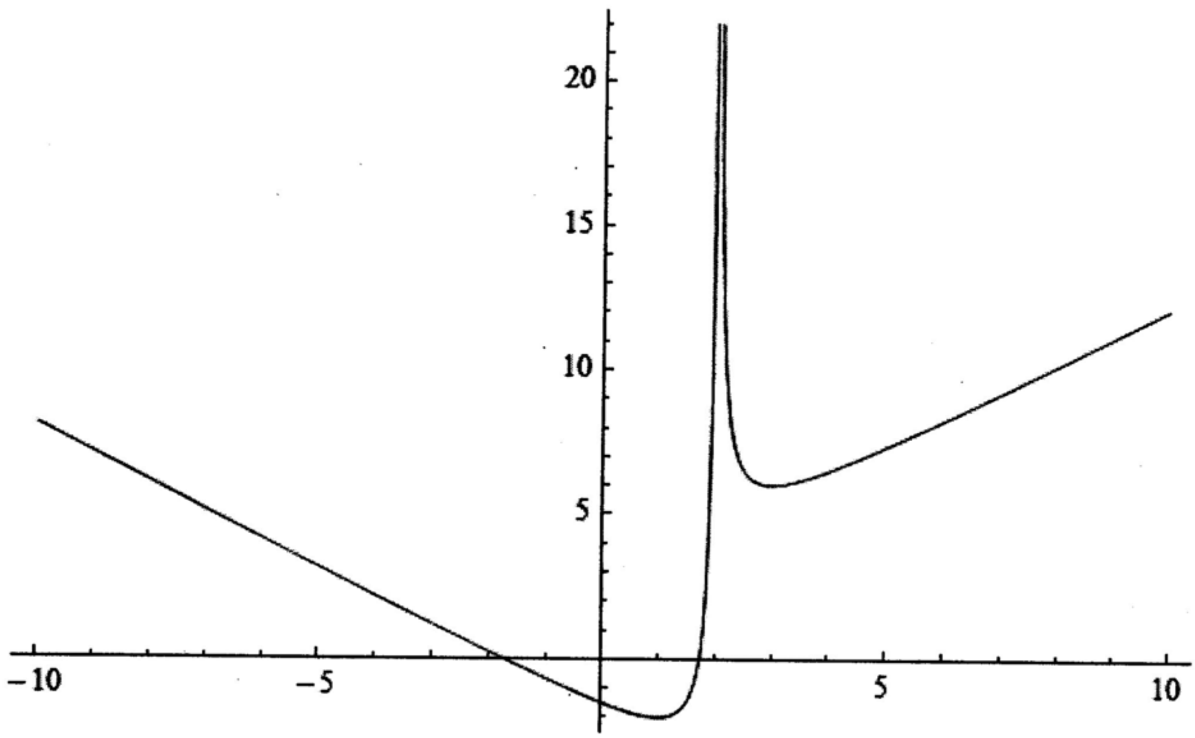
$f''(x) > 0$ except at $x=2$

$\Rightarrow f$ concave up on $(-\infty;2) \cup (2;\infty)$(!)

$\Rightarrow$ no inflection point

Graph of $f(x)$:





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