QUESTION 1 The blanks below will be filled by students. (Except the score)

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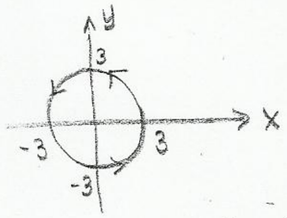
For the solution of this question please use only the front face and if necessary the back face of this page.

[6 pts] a) Find the parametric equations and a parameter interval for the motion of a particle that starts at $(3,0)$ and traces the circle $x^2+y^2=9$ twice counterclockwise (↺).

[10 pts] b) $\lim\limits_{x\to 0}\dfrac{2\sin(2x)-\sin(4x)}{x^3}=?$ (Do not use the L'Hopital's Rule).

[9 pts] c) $\lim\limits_{x\to\infty}\dfrac{\tan^{-1}(-x)+\sin^{-1}(\frac{-x}{x+1})}{\cos^{-1}(\frac{-x}{x+1})}=?$

a)



$x=3\cos t$ $\quad 0\le t\le 4\pi$

$y=3\sin t$

b) $$\lim_{x\to 0}\frac{2\sin(2x)-\sin(4x)}{x^3}=\lim_{x\to 0}\frac{2\sin(2x)-2\sin(2x)\cos(2x)}{x^3}$$

$$=\lim_{x\to 0}\frac{2\sin(2x)\,[1-\cos(2x)]}{x^3}$$

$$=\lim_{x\to 0}\frac{2\sin(2x)}{\frac{1}{2}\cdot 2x}\cdot\frac{2\sin^2 x}{x^2}$$

$$=\lim_{x\to 0} 8\cdot\frac{\sin(2x)}{2x}\cdot\left(\frac{\sin x}{x}\right)^2=8.1.1=8$$

c) $$\lim_{x\to\infty}\frac{\tan^{-1}(-x)+\sin^{-1}\left(\frac{-x}{x+1}\right)}{\cos^{-1}\left(\frac{-x}{x+1}\right)}=\frac{-\frac{\pi}{2}-\frac{\pi}{2}}{\pi}=-1$$

$$\left(\lim_{x\to\infty}\frac{-x}{x+1}=-1\right)$$

QUESTION 2 The blanks below will be filled by students. (Except the score)

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[14 pts] a) At what points is the function $f(x) = \dfrac{2}{1 - \dfrac{1}{x-1}}$ discontinuous? Give reasons for your answer.

If any, classify the types of the discontinuities.

[11 pts] b) By using Intermediate Value Theorem, show that the graphs of the functions $f(x) = x^4 - 5x^2$ and $g(x) = 2x^3 - 4x + 6$ intersect at a point between $x = 3$ and $x = 4$.

(Hint: Define $h(x) = f(x) - g(x)$)

a) $f(x) = \dfrac{2}{1-\dfrac{1}{x-1}} = \dfrac{2(x-1)}{x-2}, \; x \neq 1$, $f(x)$ is not continuous at $x=1$ and $x=2$. Because $f(x)$ is not defined at $x=1$ and $x=2$.

$\underline{x=1}$

$\lim\limits_{x\to 1^+} \dfrac{2}{1-\dfrac{1}{x-1}} = \lim\limits_{x\to 1^+} \dfrac{2(x-1)}{x-2} = 0$

$\lim\limits_{x\to 1^-} \dfrac{2}{1-\dfrac{1}{x-1}} = \lim\limits_{x\to 1^-} \dfrac{2(x-1)}{x-2} = 0$

Since $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^-} f(x)$ $f(x)$ has a removable discontinuity at $x=1$.

$\underline{x=2}$ $\lim\limits_{x\to 2^+} \dfrac{2}{1-\dfrac{1}{x-1}} = \lim\limits_{x\to 2^+} \dfrac{2(x-1)}{x-2} = +\infty$

since at least one of the one sided limit goes to ∞ $f(x)$ has an infinite discontinuity at $x=2$

(or $\lim\limits_{x\to 2^-} \dfrac{2}{1-\dfrac{1}{x-1}} = \lim\limits_{x\to 2^-} \dfrac{2(x-1)}{x-2} = -\infty$ $f(x)$ has an infinite discontinuity at $x=2$)

b) Let $h(x) = x^4 - 5x^2 - 2x^3 + 4x - 6$. Then $h(x)$ is continuous on $[3,4]$ since polynomials are continuous everywhere.

$h(3) = -12 < 0$
$h(4) = 58 > 0$
By the Intermediate Value Theorem there exists at least one $c \in [3,4]$ such that

$h(c) = 0 \Rightarrow h(c) = f(c) - g(c) = 0 \Rightarrow f(c) = g(c) \quad c \in [3,4]$

In other words, $f(x)$ and $g(x)$ intersect at a point between $x=3$ and $x=4$

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QUESTION 3 The blanks below will be filled by students. (Except the score)

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[12 pts] a) Find the equation of the tangent line to the graph of, $x\sin(x+y)=x^2-1$ at the point $(1,-1)$. (Use implicit differentiation.)

[13 pts] b) (i) State the Mean Value Theorem.

(ii) Let $f(x)$ and $g(x)$ be two functions, differentiable on the interval $[0,10]$ satisfying the relationships $f'(x)=g(x)$ and $f''(x)=-f(x)$.

Let $h(x)=f^2(x)+g^2(x)$. If $h(0)=5$, find $h(10)$.

a) Differentiate both sides of the eq. $x\sin(x+y)=x^2-1$:

$$1\cdot\sin(x+y)+x\cdot(1+y')\cos(x+y)=2x$$

At the point $(1,-1)$ substitute $x=1$ and $y=-1$,

$$1\cdot\sin(0)+1\left(1+y'\big|_{(1,-1)}\right)\cos(0)=2 \Rightarrow y'\big|_{(1,-1)}=1=\text{slope of the tangent line}$$

The eq. of the tangent line to the given curve at $(1,-1)$ is $y+1=1(x-1) \Rightarrow y=x-2$

b) i) if a function $f(x)$ is

- continuous at every point of the closed interval $[a,b]$ and
- differentiable at every point of its interior (a,b).

Then there is at least one point $c\in(a,b)$ such that

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$

ii) Since $f(x)$ and $g(x)$ are differentiable on $[0,10]$, then they must also be continuous on $[0,10]$. Therefore, $h(x)$ is both continuous and differentiable on $[0,10]$. By the Mean Value Theorem there is at least one point $c\in(0,10)$ such that

$$\frac{h(10)-h(0)}{10-0}=h'(c)=2f(c)f'(c)+2g(c)g'(c)$$

$$=2f(c)g(c)-2g(c)f(c)=0 \Rightarrow \frac{h(10)-5}{10}=0$$

$\left.\begin{array}{l}f'(x)=g(x)\\ f''(x)=g'(x)=-f(x)\end{array}\right\}$ for every $x\in[0,10]$

$h(10)=5$

Since $c\in(0,10)$, then $f'(c)=g(c)$ and $g'(c)=-f(c)$

OR $h'(x)=0 \Rightarrow$ h is constant (on $[0,10]$) by a corollary of MVT Hence, $h(10)=h(0)=5$

QUESTION 4 The blanks below will be filled by students. (Except the score)

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[25 pts] Let $f(x) = 1 + \frac{1}{x} - \frac{1}{x^2}$.

i) Find the domain and, if any, the intercepts of $f(x)$.

Domain: $(-\infty, 0) \cup (0, \infty)$ $\qquad f(x) = \frac{x^2+x-1}{x^2} = \frac{(x+1/2)^2 - 5/4}{x^2}$

x-intercepts: $\left(\frac{-1 \mp \sqrt{5}}{2}, 0\right)$

y-intercept: NO

ii) If any, find the horizontal asymptotes of $f(x)$.

$\lim_{x\to\infty} 1 + \frac{1}{x} - \frac{1}{x^2} = 1$ the line $y=1$ is a horizontal asymptote

$\lim_{x\to-\infty} 1 + \frac{1}{x} - \frac{1}{x^2} = 1$

iii) If any, find the vertical asymptotes of $f(x)$.

$\lim_{x\to0^+} \frac{x^2+x-1}{x^2} = -\infty$ the line $x=0$ is a vertical asymptote

$\lim_{x\to0^-} \frac{x^2+x-1}{x^2} = -\infty$

iv) If any, find the oblique asymptotes of $f(x)$. $y = mx+n$, $m = \lim_{|x|\to\infty} \frac{f(x)}{x} = \lim_{|x|\to\infty} \frac{x^2+x-1}{x^3} = 0$

There is no oblique asymptote.

v) Find the intervals on which the function is increasing and decreasing, and identify the function's local extreme values, if any, saying where they are taken on.

$f'(x) = -\frac{1}{x^2} + \frac{2}{x^3} = \frac{2-x}{x^3}$

	0		2	
$2-x$	+	+		−
x^3	−	+		+
$f'(x)$	−	+		−
	↘	↗		↘

$f(x)$ is decreasing in $(-\infty, 0)$ and $[2, \infty)$

$f(x)$ is increasing in $(0, 2]$

$f(x)$ has a local maximum at $x=2$

$f(2) = 5/4$

$0 \notin$ Domain $\quad x=0$ is not an extremum point

vi) Identify the concavity and, if any, find the points of inflection.

$f''(x) = \left(\frac{-1}{x^2}\right)' + \left(\frac{2}{x^3}\right)' = \frac{2}{x^3} - \frac{6}{x^4} = \frac{2x-6}{x^4}$ $<$ $f''=0$ at $x=3$; f'' is undef at $x=0 \notin$ Domain

		0		3	
$f''(x)$	−		−	0	+
	concave down		concave down		concave up.

$(-\infty,0)\cup(0,3]$ concave down

$[3,\infty)$ concave up

$x=3$ is an inf. point $\quad f(3) = \frac{11}{9}$

By using all obtained above, graph the curve of $y = f(x)$ on the back of your paper.

x	$-\infty$	$\frac{-1-\sqrt{5}}{2}$		0		$\frac{-1+\sqrt{5}}{2}$		2		3		∞
f'	$-$		$-$	‖		$+$	$+$	0	$-$		$-$	
$f(x)$	1	$\searrow$ 0	$\searrow$ $-\infty$	‖	$-\infty$ $\nearrow$	0	$\nearrow$	$\frac{5}{4}$	$\searrow$	$\frac{11}{9}$	$\searrow$	1
f''