MATHEMATICS 101E — MIDTERM — NOVEMBER 19, 2011

QUESTION 1 The blanks below will be filled by students. (Except the score)

Surname:	Name:	Group Number:	List Number:	**Score**
Signature:	Electronic Post(e-mail) address:	Student Number:		

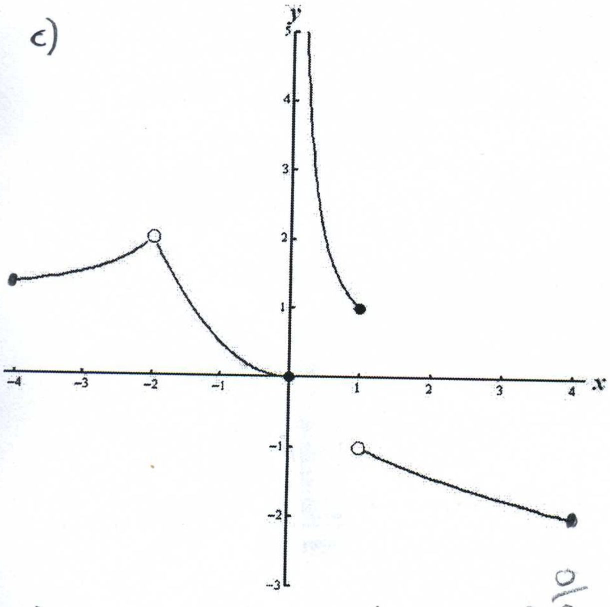
For the solution of this question please use only the front face and if necessary the back face of this page.

[5 pt] a) Graph the function of $\mathbf{f(x) = e^{x+1} - 2}$ by using horizontal and vertical shifts.

[10 pt] b) $\lim\limits_{x\to4} \dfrac{(\sqrt{x^2-7} - x + 1)\sin(x-4)}{(x^2-3x-4)^2}$ =? (Do not use the L'Hopital's Rule)

[10 pt] c) Let $\mathbf{f(x)}$ be the function illustrated in the figure below. For what values of $\mathbf{x}$ is $\mathbf{f(x)}$ discontinuous on $[-4, 4]$? Classify the types of discontinuities. Give reasons for your answer.

c)



f is discontinuous at $x=-2, 0$ and 1.

at $x=-2$: $\lim\limits_{x\to-2^-} f(x) = 2$, $\lim\limits_{x\to-2^+} f(x) = 2$

f has a removable discontinuity at $x=-2$

at $x=0$ $\lim\limits_{x\to0^-} f(x) = 0$, but $\lim\limits_{x\to0^+} f(x) = \infty$, therefore,

f has an infinite discontinuity at $x=0$

at $x=1$, since $\lim\limits_{x\to1^-} f(x) = 1 \neq \lim\limits_{x\to1^+} f(x) = -1$,

f has a jump discontinuity at $x=1$.

b) $\lim\limits_{x\to4} \dfrac{(\sqrt{x^2-7} - (x-1))\sin(x-4)}{(x-4)^2(x+1)^2} \overset{0/0}{=} \lim\limits_{x\to4}\dfrac{\sin(x-4)}{(x-4)} \cdot \lim\limits_{x\to4} \dfrac{[\sqrt{x^2-7}-(x-1)][\sqrt{x^2-7}+(x-1)]}{(x-4)(x+1)^2[\sqrt{x^2-7}+(x-1)]}$

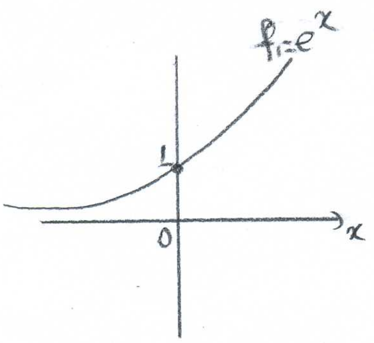
$= \underbrace{\lim\limits_{x\to4} \dfrac{\sin(x-4)}{(x-4)}}_{1} \cdot \lim\limits_{x\to4} \dfrac{x^2-7-(x^2-2x+1)}{(x-4)(x+1)^2[\sqrt{x^2-7}+(x-1)]} = \lim\limits_{x\to4} \dfrac{2\cancel{(x-4)}}{\cancel{(x-4)}(x+1)^2[\sqrt{x^2-7}+(x-1)]}$ 3 3

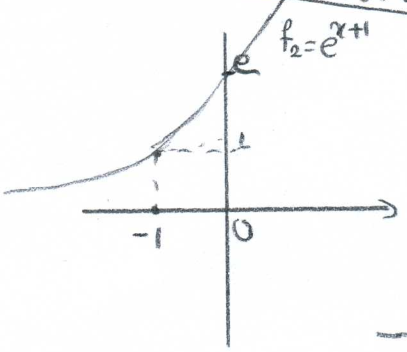
$= \dfrac{2}{25\cdot6} = \dfrac{1}{75}$ //

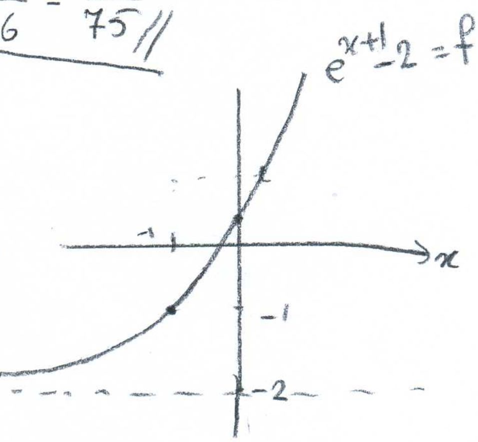
a) $f_1 = e^x$

$f_2 = e^{x+1}$

$f = e^{x+1} - 2$







MATHEMATICS 101E — MIDTERM — NOVEMBER 19, 2011

QUESTION 2 — The blanks below will be filled by students. (Except the score)

Surname:	Name:	Group Number:	List Number:	**Score**
Signature:	Electronic Post(e-mail) address:	Student Number:		

For the solution of this question please use only the front face and if necessary the back face of this page.

[10 pt] a) Show that the function $\mathbf{f(x)} = \frac{1}{\mathbf{x}}$ satisfies the Mean Value Theorem on the interval $[\mathbf{a}, \mathbf{b}]$ such that $\mathbf{a}, \mathbf{b} > 0$ and find the value of $\mathbf{c}$ in the conclusion of the theorem.

[15 pt] b) Find the equation of the normal line of the curve $\sin(2xy) + y^2 - 2xy - \tan(\pi x) = 1$ at the point $\mathbf{P(0,1)}$.

a) The Mean Value Theorem:
If $y=f(x)$ i) is continuous on $[a,b]$
ii) is differentiable in (a,b)
then, there is at least one point c in (a,b) at which $f'(c) = \frac{f(b)-f(a)}{b-a}$

$f(x) = \frac{1}{x}$, $a,b>0$

i) Is f continuous on $[a,b]$
f is undefined at $x=0 \notin [a,b] \Rightarrow f$ is con.

ii) Is f differentiable in (a,b)?
Yes, $f'(x) = \frac{-1}{x^2}$, $x \neq 0$

There is at least one point c in (a,b);

$$f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow -\frac{1}{c^2} = \frac{\frac{1}{b}-\frac{1}{a}}{b-a} \Rightarrow -\frac{1}{c^2} = \frac{\overset{(-1)}{\cancel{a-b}}}{ab\cancel{(b-a)}} \Rightarrow c^2 = ab$$

$c = \sqrt{ab}$ $\quad$ $\cancel{c = -\sqrt{ab}}$

$c \in (a,b)$, $0<a<b$.

b) $\sin(2xy) + y^2 - 2xy - \tan(\pi x) = 1$, $y = y(x)$

$\Rightarrow (2xy)' \cdot \cos(2xy) + 2yy' - 2\cdot(xy)' - \pi \sec^2(\pi x) = 0$

$\Rightarrow 2[y + xy']\cos(2xy) + 2yy' - 2[y+xy'] - \pi \sec^2(\pi x) = 0$

$\Rightarrow y'[2x\cos(2xy) + 2y - 2x] = \pi \sec^2(\pi x) - 2y\cos(2xy) + 2y$

$$y' = \frac{\pi \sec^2(\pi x) - 2y\cos(2xy) + 2y}{2x\cos(2xy) + 2y - 2x} \Rightarrow m_{tangent} = y'\Big|_{P(0,1)} = \frac{\pi \sec^2 0 - 2\cdot\cos 0 + 2}{0 + 2 - 0} = \frac{\pi}{2}$$

$$m_{normal} = -\frac{1}{m_{tangent}} = -\frac{1}{\pi/2} = -\frac{2}{\pi}$$

The normal line equation: $y - y_0 = m_{normal}(x - x_0)$

$$y - 1 = -\frac{2}{\pi}(x - 0) \Rightarrow y = \frac{-2}{\pi}x + 1$$

MATHEMATICS 101E — MIDTERM — NOVEMBER 19, 2011

QUESTION 3 The blanks below will be filled by students. (Except the score)

Surname:	Name:	Group Number:	List Number:	Score
Signature:	Electronic Post(e-mail) address:	Student Number:		

For the solution of this question please use only the front face and if necessary the back face of this page.

[8 pt] a) Let the values $\mathbf{f(2)} = \frac{\pi}{2}$, $\mathbf{g(2)} = -1$, $\mathbf{f'(2)} = 4$ and $\mathbf{g'(2)} = 0$ be given for two differentiable functions $\mathbf{f(x)}$ and $\mathbf{g(x)}$. Find $\mathbf{h'(2)}$ for the function $\mathbf{h(x) = 1 + \cos[f(x)g(x)]}$.

[17 pt] b) Find the asymptotes of the function (if any) $\mathbf{f(x)} = \dfrac{\mathbf{5x^2 + 8x - 3}}{\mathbf{x^2 - 1}}$. Give reasons for your answer.

a) $h(x) = 1 + \cos(f(x).g(x)) \Rightarrow \frac{dh(x)}{dx} = h'(x) = 0 + \frac{d}{dx}[f(x).g(x)].[-\sin(f(x).g(x))]$

$\Rightarrow h'(x) = -(f'(x).g(x) + f(x).g'(x))\sin(f(x).g(x))$

$\Rightarrow h'(2) = -(f'(2).g(2) + f(2).g'(2)).\sin(f(2).g(2)) \Rightarrow h'(2) = -(4.(-1) + \frac{\pi}{2}.0).\underbrace{\sin\left(\frac{\pi}{2}.(-1)\right)}_{(-1)}$

$= -4$ //

b) $f(x) = \dfrac{5x^2+8x-3}{x^2-1}$ $\quad$ f is not defined at $x = \pm 1$.

For the vertical asymptotes:

$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} \dfrac{5x^2+8x-3}{(x-1)(x+1)} = \dfrac{5+8-3}{-0.2} = \dfrac{10}{-0} = -\infty$

$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} \dfrac{5x^2+8x-3}{(x-1)(x+1)} = \dfrac{5+8-3}{0.2} = \dfrac{10}{0} = \infty$

$\boxed{x = 1}$ V.A.

$\lim_{x\to-1^-} f(x) = \lim_{x\to-1^-} \dfrac{5x^2+8x-3}{(x-1)(x+1)} = \dfrac{5-8-3}{-2.-0} = \dfrac{-6}{0} = -\infty$

$\lim_{x\to-1^+} f(x) = \lim_{x\to-1^+} \dfrac{5x^2+8x-3}{(x-1)(x+1)} = \dfrac{5-8-3}{-2.0} = \dfrac{-6}{-0} = \infty$

$\boxed{x = -1}$ V.A.

For the horizontal asymptotes: $(\deg(5x^2+8x-3) = \deg(x^2-1))$

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{5x^2+8x-3}{x^2-1} = \lim_{x\to\infty} \dfrac{\cancel{x^2}(5+8/x-3/x^2)}{\cancel{x^2}(1-1/x^2)} = \dfrac{5}{1} = 5$

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \dfrac{(5+8/x-3/x^2)}{1-1/x^2} = \dfrac{5}{1} = 5$

$\boxed{y = 5}$ H.A.

No oblique asymptotes.

MATHEMATICS 101E MIDTERM NOVEMBER 19, 2011

QUESTION 4 The blanks below will be filled by students. (Except the score)

Surname:	Name:	Group Number:	List Number:	Score
Signature:	Electronic Post(e-mail) address:	Student Number:		

For the solution of this question please use only the front face and if necessary the back face of this page.

[25 p] For the function $\mathbf{f(x) = x\sqrt{4-x^2}}$,
i) find the domain ~~and range~~
ii) find the intervals on which the function is increasing and decreasing,
iii) find the extrema and where they occur,
iv) identify the concavity and, if any, find the points of inflection,
v) sketch the graph.

i) its domain: $4-x^2 \geq 0 \Rightarrow |x| \leq 2$ its range: Since $-2 \leq x \leq 2$, $f(x) \in [-2,2]$
$D = \{x \in \mathbb{R} \mid -2 \leq x \leq 2\}$

ii) $f' = ?$ $f'(x) = 1.(4-x^2) + x.(-2x).\frac{1}{2(4-x^2)^{1/2}} = \frac{4-x^2-x^2}{\sqrt{4-x^2}} \Rightarrow f'(x) = \frac{2(2-x^2)}{\sqrt{4-x^2}}$,

$f'(x) = 0 \Rightarrow x = \pm\sqrt{2} \in D$.

x	-2		$-\sqrt{2}$		$\sqrt{2}$		2
$2(2-x^2)$		$-$	0	$+$	0	$-$	
$\sqrt{4-x^2}$		$+$		$+$		$+$	
f'		$-$		$+$		$-$	
		dec.	local min	inc.	local max	dec.	

f is decreasing on $[-2,-\sqrt{2}) \cup (\sqrt{2},2]$
f " increasing in $[-\sqrt{2},\sqrt{2}]$

iii) The critical points are $x = -\sqrt{2}, \sqrt{2}$ (f' is zero) and f' changes its sign at these points.
The end points are $x = -2, 2$.
$f(-2) = 0$ $f(-\sqrt{2}) = -2 \rightarrow$ local minima (abs. minima)
$f(2) = 0$ $f(\sqrt{2}) = 2 \rightarrow$ local maxima (abs. maxima)

iv) $f'' = ?$ $f'' = \frac{d}{dx}\left[\frac{2(2-x^2)}{\sqrt{4-x^2}}\right] = 2\left\{(-2x)(4-x^2)^{1/2} - (2-x^2)\frac{(-2x)}{2\sqrt{4-x^2}}\right\}\frac{1}{(4-x^2)} = 2\,\frac{x(x^2-6)}{(4-x^2)^{3/2}}$

$f'' = 0 \Rightarrow x = 0 \in D$, $x = \pm\sqrt{6} \notin D$

x	-2		0		2
x		$-$		$+$	
x^2-6		$-$		$-$	
$(4-x^2)^{3/2}$		$+$		$+$	
f''		$+$	inf. point	$-$	
		conc. up		conc. down	

f is concave up over $(-2,0)$
f " " down over $(0,2)$
$f(0) = 0$ is the inf. point.

v)

x	-2		$-\sqrt{2}$		0		$\sqrt{2}$		2
f'		$-$		$+$		$+$		$-$	
f''		$+$		$+$		$-$		$-$	
f	0		-2		0		2		0