QUESTION 1 The blanks below will be filled by the students. (Except the score)

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For the solution of this question please use only the front face and if necessary the back face of this page.

[10 pt] a) Let $\mathbf{f(x) = 1 + \sin^{-1}(e^x - 2)}$. Find the domain of $\mathbf{f(x)}$. Show that $\mathbf{f(x)}$ is one-to-one, and then find its inverse.

[8 + 7 pt] b) Find the following limits if exists (do not use L'Hopital's rule):

$$i)\quad \lim_{x\to 2} \frac{1}{3 - 4^{\frac{1}{x-2}}} \qquad ii)\quad \lim_{x\to\infty} x\sin\left(\frac{2-3x}{1+x^2}\right).$$

a) $|e^x - 2| \leq 1 \Rightarrow -1 \leq e^x - 2 \leq 1 \Rightarrow 1 \leq e^x \leq 3 \Rightarrow \underline{0 \leq x \leq \ln 3}.$

$f(x_1) = f(x_2) \Rightarrow 1 + \sin^{-1}(e^{x_1} - 2) = 1 + \sin^{-1}(e^{x_2} - 2)$

$\Rightarrow \sin^{-1}(e^{x_1} - 2) = \sin^{-1}(e^{x_2} - 2).$

Since $\sin^{-1}$ is 1-1, we have $e^{x_1} - 2 = e^{x_2} - 2 \Rightarrow x_1 = x_2$

$\Rightarrow f(x)$ is 1-1.

$y = 1 + \sin^{-1}(e^x - 2) \Rightarrow \sin(y-1) = e^x - 2 \Rightarrow e^x = 2 + \sin(y-1)$

$\Rightarrow x = \ln(2 + \sin(y-1)).$ $x \leftrightarrow y$, $\underline{f^{-1}(x) = \ln(2 + \sin(x-1))}$

b) (i)

$$\left.\begin{array}{l} \lim\limits_{x\to 2^+} \dfrac{1}{3 - 4^{1/x-2}} = \dfrac{1}{3 - 4^{\infty}} = 0 = L_1 \\[2ex] \lim\limits_{x\to 2^-} \dfrac{1}{3 - 4^{1/x-2}} = \dfrac{1}{3 - 4^{-\infty}} = \dfrac{1}{3-0} = \dfrac{1}{3} = L_2 \end{array}\right\} \Rightarrow L_1 \neq L_2$$

So $\lim\limits_{x\to 0} \dfrac{1}{3 - 4^{1/x-2}}$ does not exist.

(ii)
$$\underset{(\infty\cdot 0)}{\lim_{x\to\infty} x\sin\left(\frac{2-3x}{1+x^2}\right)} = \lim_{x\to\infty} \frac{\sin\left(\frac{2-3x}{1+x^2}\right)}{1/x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{2-3x}{1+x^2}}{\frac{1}{x}} \cdot \lim_{x\to\infty} \frac{\sin\left(\frac{2-3x}{1+x^2}\right)}{\left(\frac{2-3x}{1+x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{2x - 3x^2}{1+x^2} \cdot 1 = -3.$$

QUESTION 2 The blanks below will be filled by students. (Except the score)

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[13 pt] a) By using the definition of derivative show that whether the function $\mathbf{f(x) = |x| \cos x}$ is differentiable at $\mathbf{x = 0}$ or not.

[6+6 pt] b) i) $\mathbf{f(x) = \sec\left(\frac{2}{x}\right) + \frac{\sqrt{x}}{x+1}}$; $\mathbf{f'(x)}$ =? ii) $\mathbf{f(x) = \frac{1+\csc x}{x+\cot x}}$; $\mathbf{f'\left(\frac{\pi}{4}\right)}$ =?

a) $f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|0+h|\cos(0+h) - 0}{h}$

$= \lim_{h\to 0} \frac{|h|}{h}\cosh$

Right-derivative: $f'_+(0) = \lim_{h\to 0^+} \frac{|h|}{h}\cos h = \lim_{h\to 0^+} \frac{h\cos h}{h} = 1$

Left-derivative: $f'_-(0) = \lim_{h\to 0^-} \frac{|h|}{h}\cos h = \lim_{h\to 0^-} \frac{-h\cos h}{h} = -1$

As $f'_+(0) \neq f'_-(0)$, $f(x)$ has no derivative at 0.

b-i) $f'(x) = -\frac{2}{x^2}\sec\left(\frac{2}{x}\right)\tan\left(\frac{2}{x}\right) + \frac{\frac{x+1}{2\sqrt{x}} - \sqrt{x}}{(x+1)^2}$

$= -\frac{2}{x^2}\sec\left(\frac{2}{x}\right)\tan\left(\frac{2}{x}\right) + \frac{1-x}{2\sqrt{x}(1+x)^2}$

ii) $f'(x) = \frac{-\csc x\cot x\,(x+\cot x) - (1+\csc x)(1-\csc^2 x)}{(x+\cot x)^2}$

$f'\left(\frac{\pi}{4}\right) = \frac{-\sqrt{2}\cdot 1\cdot\left(\frac{\pi}{4}+1\right) - (1+\sqrt{2})(1-(\sqrt{2})^2)}{\left(\frac{\pi}{4}+1\right)^2} = \frac{16-4\pi\sqrt{2}}{(4+\pi)^2}$

QUESTION 3 The blanks below will be filled by the students. (Except the score)

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[15pt] a) State the Mean Value Theorem, and then prove it.

[10pt] b) Does the function $\mathbf{f(x) = \sqrt{2x+1} + x}$ satisfy the hypotheses of Mean Value Theorem on the interval $[\mathbf{0},\mathbf{1}]$? If so, find a number $\mathbf{c}$ that satisfies the theorem. Give reasons for your answers.

(a) The Mean Value Theorem: Suppose that $y=f(x)$ is continuous on a closed interval $[a,b]$ and differentiable on the interval's interior (a,b). Then there is at least one point $c\in(a,b)$ at which

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

Proof: The equation of the line through $A(a,f(a))$ and $B(b,f(b))$ is

$$y=g(x)=\frac{f(b)-f(a)}{b-a}(x-a)+f(a)$$

which is a linear function in x. It is continuous and differentiable for all x. Put $h(x)=f(x)-g(x)$. It is continuous on $[a,b]$ and differentiable on (a,b) as f and g are. Since

$$h(a)=f(a)-g(a)=f(a)-\left[0+f(a)\right]=0 \quad \text{and}$$

$$h(b)=f(b)-g(b)=f(b)-\left[\frac{f(b)-f(a)}{b-a}(b-a)+f(a)\right]=0,$$

then $h(x)$ holds the hypotheses of Rolle's theorem. Thus $h'=0$ at some $c\in(a,b)$ That is

$$0=h'(c)=f'(c)-g'(c)=f'(c)-\left(\frac{f(b)-f(a)}{b-a}\right)$$

$$\Rightarrow f'(c)=\frac{f(b)-f(a)}{b-a}.$$

(b) $f(x)=\sqrt{2x+1}+x$. $D(f): [-\frac{1}{2},\infty)$. Since $\sqrt{2x+1}$ and x are continuous on $D(f)$, then $f(x)$ is continuous on $[0,1]\subset D(f)$. $f'(x)=\frac{2}{2\sqrt{2x+1}}+1$ which exists on $(0,1)$.
That is, $f(x)$ is differentiable on $(0,1)$.

Therefore $f(x)$ satisfy the hypotheses of M.V.T. So, there exists at least one $c\in(0,1)$ at which

$$f'(c)=\frac{1}{\sqrt{2c+1}}+1=\frac{f(1)-f(0)}{1-0}=\frac{\sqrt{3}+1-1}{1}$$

$$\Rightarrow \sqrt{2c+1}=\frac{1}{\sqrt{3}-1}=\frac{\sqrt{3}+1}{2}\Rightarrow 2c=\frac{(\sqrt{3}+1)^2}{4}-1$$

$$c=\frac{1}{2}\left(\frac{(\sqrt{3}+1)^2-4}{4}\right)=\frac{\sqrt{3}}{4}\in(0,1).$$

QUESTION 4 The blanks below will be filled by the students. (Except the score)

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Let $\mathbf{f(x)} = \sqrt{3}\mathbf{x} + \cos(\mathbf{2x}), \quad \mathbf{x} \in [\mathbf{0}, \pi]$.

[6 pt] (a) Find the critical points and critical values of $\mathbf{f(x)}$.

[7 pt] (b) Using the second derivative test to find the local extreme values of $\mathbf{f(x)}$ if exists.

[6 pt] (c) Find the absolute extreme values of the function on the interval.

[6 pt] (d) Determine the concavity of the function on the interval.

a) $f'(x) = \sqrt{3} - 2\sin(2x) = 0 \Rightarrow 2x = \frac{\pi}{3} + 2\pi k$ or $2x = \frac{2\pi}{3} + 2\pi k$, $k = 0, \mp 1, \mp 2, \ldots$

$\Rightarrow x = \frac{\pi}{6}$, $x = \frac{\pi}{3} \in [0, \pi]$ are critical points in the interval

Critical values: $f(\frac{\pi}{6}) = \sqrt{3}\frac{\pi}{6} + \frac{1}{2}$, $f(\frac{\pi}{3}) = \sqrt{3}\frac{\pi}{3} - \frac{1}{2}$.

b) $f''(x) = -4\cos(2x)$. At $x = \frac{\pi}{6}$ $f''(\frac{\pi}{6}) = -4 \cdot \cos(\frac{\pi}{3}) = -2 < 0$.

So, $f(x)$ has a local maximum value $f(\frac{\pi}{6}) = \frac{\sqrt{3}\pi}{6} + \frac{1}{2}$ at $x = \frac{\pi}{6}$.

At $x = \frac{\pi}{3}$ $f''(\frac{\pi}{3}) = -4\cos(\frac{2\pi}{3}) = 2 > 0$. So, $f(x)$ has a local minimum value $f(\frac{\pi}{3}) = \sqrt{3}\frac{\pi}{3} - \frac{1}{2}$ at $x = \frac{\pi}{3}$.

c) At endpoints $f(0) = 1$, $f(\pi) = \sqrt{3}\pi + 1$

Comparing $f(0) = 1$, $f(\pi) = 1 + \sqrt{3}\pi$, $f(\frac{\pi}{6}) = \frac{1}{2} + \frac{\sqrt{3}\pi}{6}$, $f(\frac{\pi}{3}) = \frac{\sqrt{3}\pi}{3} - \frac{1}{2}$

the absolute maximum value is $f(\pi) = 1 + \sqrt{3}\pi$, and the absolute minimum value is $f(0) = 1$

d) $f''(x) = -4\cos(2x) = 0 \Rightarrow 2x = \mp\frac{\pi}{2} + 2\pi k$, $k = 0, \mp 1, \ldots$

$x = \frac{\pi}{4}, \frac{3\pi}{4} \in [0, \pi]$

are the inflection points of the function $f(x)$ on $[0, \pi]$.

$f''(x) < 0$ on $(0, \frac{\pi}{4})$, $f''(x) > 0$ on $(\frac{\pi}{4}, \frac{3\pi}{4})$

and $f''(x) < 0$ on $(\frac{3\pi}{4}, \pi)$.

$\Rightarrow$ The graph of f is concave down on $(0, \pi/4)$, concave up on $(\pi/4, \frac{3\pi}{4})$ and concave down on $(\frac{3\pi}{4}, \pi)$.

$$\mathrm{Plot}\left[\sqrt{3}x + \mathrm{Cos}[2x], \{x, 0, \pi\}, \mathrm{PlotStyle} \to \mathrm{Thickness}[0.01]\right]$$