QUESTION 1 The blanks below will be filled by students. (Except the score)

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For the solution of this question please use only the front face and if necessary the back face of this page.

[12 pt] a) Find the limit: $\lim_{x\to 3}\frac{\cos(\frac{\pi}{2}x)}{(x-3)}$ =? (Do not use the L'Hopital rule).

[13 pt] b) Find all points on the curve $x^2y^2+xy=2$, where the slope of the tangent line is -1. (Use the implicit differentiation).

(a) $\lim_{x\to 3}\frac{\cos(\pi/2\, x)}{x-3} = ?$ $\quad x-3=u,\quad x\to 3,\ u\to 0$

$\lim_{u\to 0}\frac{\cos(\pi/2\,(u+3))}{u}$, $\quad \cos\frac{\pi}{2}(u+3) = \cos\frac{\pi}{2}u\ \underbrace{\cancel{\cos}\frac{3\pi}{2}}_{=0} - \underbrace{\sin\left(\frac{3\pi}{2}\right)}_{=-1}\sin(\pi/2\, u)$

$\lim_{u\to 0}\frac{\pi}{2}\cdot\underbrace{\frac{\sin(\pi/2\, u)}{\pi/2 .\, u}}_{1} = \underline{\underline{\pi/2}}$

(b) $\left.\frac{dy}{dx}\right|_{(x_0,y_0)} = -1$, $\quad 2xy^2 + x^2.\,2.y.\frac{dy}{dx} + x\frac{dy}{dx} + 1.y = 0$

$2x_0y_0^2 + 2x_0^2y_0\underbrace{\left.\frac{dy}{dx}\right|_{(x_0,y_0)}}_{=-1} + y_0 + x_0\underbrace{\left(\frac{dy}{dx}\right)_{x_0,y_0}}_{=-1} = 0$

$2x_0y_0^2 - 2x_0^2y_0 + y_0 - x_0 = 0$

$(y_0-x_0)(2x_0y_0+1)=0$ $\quad x_0=y_0$ or $x_0y_0=-1/2$

If $x_0=y_0 \Rightarrow x_0^2y_0^2 + x_0y_0 = 2$,

$x_0^4 + x_0^2 - 2 = 0$, $\quad x_0^2 = u$

$u^2 + u - 2 = 0$ — $u = -2$, $u = -2 \neq x_0^2$

$u = 1$, $u = 1 = x_0^2$

$x_0 = 1$, $\quad x_0 = -1$

$y_0 = 1$, $\quad y_0 = -1$

If $\quad x_0 y_0 = -\frac{1}{2}$, $\quad \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4} \neq 2$

A $(\underline{-1, -1})$, B $\underline{(1,1)}$

QUESTION 2 The blanks below will be filled by students. (Except the score)

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[15 pt] a) Let

$$f(x) = \begin{cases} g(x)\cos\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

where $g(x)$ is differentiable function with $g(0) = g'(0) = 0$. Show that $f(x)$ is differentiable at $x = 0$.

[10 pt] b) Find the equation of the normal line to the curve given by the parametric equations $x = 2\sec t$, $y = \sqrt{3}\tan t$, $0 < t < \frac{\pi}{4}$ at the point $t = \frac{\pi}{6}$.

(a) $f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}$

$f'(0) = \lim_{h\to 0} \frac{g(h)\cos\frac{1}{h}}{h}$

$g'(0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = 0 \Rightarrow \lim_{h\to 0} \frac{g(h)}{h} = g'(0) = 0$

$-1 \le \cos\frac{1}{h} \le 1 \Rightarrow -\frac{g(h)}{h} \le \frac{g(h)}{h}\cos\frac{1}{h} \le \frac{g(h)}{h}$

Sandwich teorem: $\lim_{h\to 0} \frac{g(h)}{h} = 0 \Rightarrow \lim_{h\to 0} \frac{g(h)}{h}\cos\frac{1}{h} = 0$

(b) $\frac{dy}{dx} = \left.\frac{dy/dt}{dx/dt}\right|_{t=\pi/6} = \left.\frac{\sqrt{3}\sec^2 t}{2\sec t \cdot \tan t}\right|_{t=\pi/6}$

$$\left.\frac{dy}{dx}\right|_{t=\pi/6} = \frac{4/\sqrt{3}}{4/3} = \sqrt{3}$$

$m_T \cdot m_N = -1 \Rightarrow m_N = -\frac{1}{\sqrt{3}}$

$x_0 = 2 \cdot \sec\pi/6 = \frac{4}{\sqrt{3}}$

$y_0 = \sqrt{3} \cdot \frac{1}{\sqrt{3}} = 1$

$y = -\frac{1}{\sqrt{3}}\left(x - \frac{4}{\sqrt{3}}\right) + 1 \Rightarrow y = -\frac{1}{\sqrt{3}}x + \frac{7}{3}$

QUESTION 3 The blanks below will be filled by students. (Except the score)

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[8 pt] a) Show that the equation $\frac{\tan x}{x} = \frac{3}{2}$ has at least one positive root.

[9 pt] b) Given that $|\mathbf{f'(x)}| < \mathbf{1}$ for all real numbers $\mathbf{x}$, show that

$$|f(x_1) - f(x_2)| < |x_1 - x_2|$$

for all real numbers $\mathbf{x_1}$ and $\mathbf{x_2}$.

[4 pt] c) Is there a function $\mathbf{f(x)}$ such that $\mathbf{f(x)}$ is not differentiable at $\mathbf{x = 0}$ and $\mathbf{(f(x))^2}$ is differentiable at $\mathbf{x = 0}$. Give reasons for your answer.

[4 pt] d) Is there a function $\mathbf{f(x)}$ such that $\mathbf{f(3) = 5}$ and $\lim_{x\to 3} \mathbf{f'(x)} = \infty$. Give reasons for your answer.

a) Let $f(x) = 2\tan x - 3x$, then

$$f\left(\frac{\pi}{3}\right) = 2\tan\frac{\pi}{3} - 3\cdot\frac{\pi}{3} = 2\sqrt{3} - \pi > 0$$

$$f\left(\frac{\pi}{4}\right) = 2 - \frac{3}{4}\pi = 2 - \frac{3}{4}\cdot 3{,}14 < 0$$

$f(x)$ is cont on $\left[\frac{\pi}{4}, \frac{\pi}{3}\right]$. By using the Intermedio theorem, there is c in $\left[\frac{\pi}{4}, \frac{\pi}{3}\right]$ such that $f(c)=0$

b) $|f'(x)| < 1$, $-1 < f'(x) < 1$

$f(x)$ is cont. on $[x_1, x_2]$

$f(x)$ is diff. on (x_1, x_2).

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}, \quad c \in (x_1, x_2)$$

$$|f(x_2) - f(x_1)| = \underbrace{|f'(c)|}_{<1}\, |x_2 - x_1| < |x_2 - x_1|$$

(c) i) $f(x) = |x|$ is not differentiable at $x=0$.
$(f(x))^2 = |x|^2 = x^2$ is differentiable at $x=0$.

ii) $f(x) = (x-3)^{1/3} + 5$, $f(3)=5$.

$$f'(x) = \frac{1}{(x-3)^{2/3}}, \quad \lim_{x\to 3} f'(x) = \infty$$

$$f'(0^+) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{|h|-|0|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$$

$$f'(0^-) = \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{|h|-|0|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$$

$f'(0^+) \neq f'(0^-)$, $f(x)=|x|$ is not diff. at $x=0$.

$$(f(x))^2 = x^2 \quad \frac{d}{dx}(x^2) = 2x, \quad \left(\frac{d}{dx}(x^2)\right)\bigg|_{x=0} = 2.0 = 0,$$

$(f(x))^2 = x^2$ is diff. at $x=0$.

QUESTION 4 The blanks below will be filled by students. (Except the score)

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[25 p] For the function $\mathbf{f(x) = \dfrac{x^2+3}{x^2-1}}$

i) find the domain and intercept points,
ii) find all asymptotes, if any,
iii) find the intervals on which the function is increasing and decreasing,
iv) find the local extrema and where they occur,
v) identify the concavity and, if any, find the points of inflection,
vi) sketch the graph.

(a) $x^2-1=0 \Rightarrow x=\mp 1$, so $D: \mathbb{R}-\{1,-1\}$

$x=0,\ y=-3,\quad y=0 \Rightarrow x^2+3 \neq 0$

(b) $\lim\limits_{x\to -1^-} \dfrac{x^2+3}{x^2-1} = \infty$, $\lim\limits_{x\to -1^+} \dfrac{x^2+3}{x^2-1} = -\infty$

$\lim\limits_{x\to 1^+} \dfrac{x^2+3}{x^2-1} = \infty$, $\lim\limits_{x\to 1^-} \dfrac{x^2+3}{x^2-1} = -\infty$

$x=\mp 1$ vertical asym.

$\lim\limits_{x\to \mp\infty} \dfrac{x^2+3}{x^2-1} = 1 \Rightarrow y=1$ H.A

(c) $f'(x) = -\dfrac{8x}{(x^2-1)^2}$, $x=0$ C.P:

$f'(x)>0,\ x<0 \Rightarrow x\in(-\infty,-1)\cup(-1,0)$, increasing

$f'(x)<0,\ x>0 \Rightarrow x\in(0,1)\cup(1,\infty)$, decreasing.

(d) $f''(x) = \dfrac{8(3x^2+1)}{(x^2-1)^3}$

$f'(0) = -8 < 0$, $f(x)$ has a local max at $x = 0$

$f(0) = 3$

		-1		1	
(x^2-1)	$+$	0	$-$	0	$+$

$f''(x) > 0$, $(-\infty, -1)$ and $(1, \infty)$ Concave up

$f''(x) < 0$ $(-1, 1)$ Concave down.

No inflection point: