MAT 101E MIDTERM 09.04.2011

QUESTION 1

Student Number:	e-mail:	Group:	List No.:	Grade
Name:	Surname:		Sign.:	

[10p]**a)** Discuss the continuity of the function $\mathbf{f(x)} = \dfrac{|1-x|}{(x^2-1)5^{\frac{|1-x|}{x}}}$. Find and classify the discontinuity points if any. Explain your answer.

b) Compute the following integrals.

[7p]**i)** $\displaystyle\int_1^{\sqrt{5}} \sqrt[4]{\frac{3x^2+15}{x^{14}}}\,dx$

[8p]**ii)** $\displaystyle\int \frac{(4x+2\sin 2x)}{\sqrt{x^2+\sin^2 x}}\,dx$

a) $x^2-1=0 \rightarrow$ $x=1$ and $x=-1$ are discontinuity points. $x=0$ is also a discontinuity point since $5^{\frac{|1-x|}{x}}$ is discontinuos at $x=0$.

$$\lim_{x\to 1^-} \frac{\cancel{(1-x)}(-1)}{\cancel{(x-1)}(x+1)\cdot 5^{\frac{1-x}{x}}} = \frac{-1}{2.5^0} = -\frac{1}{2}$$

$$\lim_{x\to 1^+} \frac{\cancel{(x-1)}.1}{\cancel{(x-1)}(x+1).5^{\frac{x-1}{x}}} = \frac{1}{2.5^0} = \frac{1}{2}$$

since $\lim\limits_{x\to 1^-} f(x) \neq \lim\limits_{x\to 1^+} f(x)$ Jump discontinuity at $x=1$.

$$\lim_{x\to -1^-} \frac{\cancel{(1+x)}.^{(-1)}}{\cancel{(x-1)}(x+1).5^{\frac{1-x}{x}}} = +\infty$$

$$\lim_{x\to -1^+} \frac{\cancel{(1-x)}^{(-1)}}{\cancel{(x-1)}(x+1)\,5^{\frac{1-x}{x}}} = -\infty$$

Infinite discontinuity at $x=-1$

$$\lim_{x\to 0^-} \frac{\cancel{(1-x)}^{(-1)}}{\cancel{(x-1)}(x+1).5^{\frac{1-x}{x}}} = -\infty$$

$$\lim_{x\to 0^+} \frac{\cancel{(1-x)}^{(-1)}}{\cancel{(x-1)}(x+1)\,5^{\frac{1-x}{x}}} = 0$$

Infinite discontinuity at $x=0$

1-b) i) $\displaystyle\int_1^{\sqrt{5}} \sqrt[4]{\frac{3x^2+15}{x^{14}}}\,dx = \int_1^{\sqrt{5}} \sqrt[4]{\frac{3x^2+15}{x^{12}x^2}}\,dx$

$$I = \int_1^{\sqrt{5}} \sqrt[4]{\frac{3x^2+15}{x^2}} \cdot \frac{1}{x^3}\,dx = \int_1^{\sqrt{5}} \sqrt[4]{3+\frac{15}{x^2}}\,\frac{1}{x^3}\,dx$$

Let $3+\dfrac{15}{x^2} = u \implies -\dfrac{30}{x^3}\,dx = du$

$$I = -\frac{1}{30}\int_{u_0}^{u_1} \sqrt[4]{u}\,du = -\frac{1}{30}\,\frac{u^{5/4}}{5/4}\Bigg|_{u_0}^{u_1} = -\frac{2}{75}\,u^{5/4}\Bigg|_{u_0}^{u_1}$$

$$I = -\frac{2}{75}\left(3+\frac{15}{x^2}\right)\Bigg|_1^{\sqrt{5}} = -\frac{2}{75}\left[(3+3)^{5/4} - (3+15)^{5/4}\right]$$

$$I = -\frac{2}{75}\left(6^{5/4} - 18^{5/4}\right) = -\frac{2}{75}\left(6\sqrt[4]{6} - 18\sqrt[4]{18}\right)$$

$$= -\frac{12}{75}\left(\sqrt[4]{6} - 3\sqrt[4]{18}\right)$$

$$= -\frac{4}{25}\left(\sqrt[4]{6} - 3\sqrt[4]{18}\right)$$

ii) $\displaystyle I = \int \frac{(4x+2\sin 2x)}{\sqrt{x^2+\sin^2 x}}\,dx = 2\int \frac{(2x+2\sin x\cos x)}{\sqrt{x^2+\sin^2 x}}\,dx$

let $x^2+\sin^2 x = u \rightarrow (2x+2\sin x\cos x)\,dx = du$

$$I = 2\int \frac{du}{u^{1/2}} = 2.2.u^{1/2} + C = 4\sqrt{u} + C = 4\sqrt{x^2+\sin^2 x} + C$$

QUESTION 2

Student Number:	e-mail:	Group:	List No.:	Grade
Name:	Surname:		Sign.:	

[10p]**a)** Find the equation of the line that is tangent to the curve

$$\mathbf{x}\cos(\mathbf{xy^2 - y}) = \frac{(\mathbf{x+y})^2}{4} \text{ at the point } \mathbf{P(1,1)}.$$

b) For the parametrized curve $\mathbf{x(t)} = \sqrt{\mathbf{t-1}}$, $\mathbf{y(t)} = \sqrt{\mathbf{2t}}$, calculate the following derivatives

[7p]**i)** $\dfrac{\mathbf{dy}}{\mathbf{dx}}$ at the point $(\mathbf{x_0, y_0}) = (\mathbf{1, 2})$,

[8p]**ii)** $\dfrac{\mathbf{d^2y}}{\mathbf{dx^2}}$ at the point $(\mathbf{x_0, y_0}) = (\mathbf{1, 2})$.

a) Derive both sides of the equation with respect to x,

$$1.\cos(xy^2-y) - x\sin(xy^2-y).(y^2+2xyy'-y') = \frac{2(x+y)}{4}.(1+y')$$

$m = y'\big|_{(1,1)}$ is the slope of the line. So substitute $(x_0,y_0)=(1,1)$ in the above equation.

$$\underbrace{\cos(1-1)}_{1} - \sin\underbrace{(\cancel{1}-1)}_{0}.(1+y'\big|_{(1,1)}) = \frac{2(1+1)}{4}.(1+y'\big|_{(1,1)})$$

$$1 = (1+y'\big|_{(1,1)}) \Rightarrow y'\big|_{(1,1)} = 0$$

$m=0$, $(x_0,y_0)=(1,1)$

The equation of the tangent line is $y-y_0 = m(x-x_0)$

$$\Rightarrow y-1=0 \Rightarrow \boxed{y=1}$$

b) $$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{dy/dt}{dx/dt} = \frac{(\sqrt{2}/2)t^{-1/2}}{(1/2)(t-1)^{-1/2}} = \sqrt{2}\sqrt{\frac{t-1}{t}}$$

For $(x_0,y_0)=(1,2)$ $\boxed{t=2}$ $\Rightarrow \left(\dfrac{dy}{dx}\right)\Big|_{t=2} = \sqrt{2}\sqrt{\dfrac{2-1}{2}} = \sqrt{2}.\dfrac{1}{\sqrt{2}} = 1$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\Big(\underbrace{\frac{dy}{dx}}_{y'}\Big) = \frac{dy'}{dt}.\frac{dt}{dx} = \frac{dy'}{dt}\Big/\frac{dx}{dt}, \left(\text{Here } y' = \frac{dy}{dx}\right)$$

$$\frac{d^2y}{dx^2} = \left[\frac{\sqrt{2}}{\cancel{2}}.\left(1-\frac{1}{t}\right)^{-1/2}\left(\frac{1}{t^2}\right)\right]\Big/\left[\frac{1}{\cancel{2}}(t-1)^{-1/2}\right] = \frac{\sqrt{2}.\cancel{\sqrt{t-1}}.\sqrt{t}}{\cancel{\sqrt{t-1}}\,t^2} = \frac{\sqrt{2}}{t^{3/2}}$$

$$\frac{d^2y}{dx^2}\Big|_{t=2} = \frac{\sqrt{2}}{2^{3/2}} = 2^{1/2}.2^{-3/2} = \frac{1}{2}$$

MAT 101E | MIDTERM | 09.04.2011

QUESTION 3

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[10p]**a)** Show that the equation $\sqrt[3]{\mathbf{x}} + \mathbf{2x} = \mathbf{1}$ has at most one root in the interval $(-\infty, \infty)$.

[15p]**b)** Let $\mathbf{f(x)}$ be a function such that $\mathbf{f(0) = 0}$ and $\mathbf{f'(x)} = \dfrac{\mathbf{x^2}}{\mathbf{1+x^4}}$ for all $\mathbf{x}$. Show that $\mathbf{0 < f(x) < x}$ for $\mathbf{x > 0}$.

a) $f(x) = \sqrt[3]{x} + 2x - 1$. Since $f(x)$ is continuous on $(-\infty, \infty)$, then $f(x)$ is continuous on $[0,1]$.

Then $f(0) = -1 < 0$ and $f(1) = 2 > 0$.

By Intermediate Value Theorem, we conclude that $f(x)$ has at least one root in $(0,1)$. That is, there is one $x_0 \in (0,1)$ at which

$f(x_0) = \sqrt[3]{x_0} + 2x_0 - 1 = 0$ also, $f'(x) = \frac{1}{3}\frac{1}{x^{2/3}} + 2 > 0$

that is, $f(x)$ is increasing on $(0,1)$.

Therefore, the root x_0 is unique on $(-\infty, \infty)$.

b) Since $f'(x)$ exists everywhere, $f(x)$ is a continuous function. Therefore, $f(x)$ is continuous on $[0,x]$ and $f'(x)$ exists on $(0,x)$. Then we can apply the Mean Value Theorem to $f(x)$.

There exists $c \in (0,x)$ such that $f'(c) = \dfrac{f(x) - \overbrace{f(0)}^{=0}}{x - 0}$

$f'(c) = \dfrac{c^2}{1+c^4} = \dfrac{f(x)}{x}$ $\quad 0 < \dfrac{c^2}{1+c^4} < \dfrac{c^4}{1+c^4} < 1$, For $c > 1$

$\Rightarrow$ $\boxed{\text{For } c > 1,\ 0 < f'(c) < 1}$

For $c < 1$, $0 < \dfrac{c^2}{1+c^4} < \dfrac{c^2}{1+c^2} < 1 \Rightarrow \boxed{\text{For } c < 1,\ 0 < f'(c) < 1}$

Therefore, in any case $0 < f'(c) < 1$, substitute $f'(c) = \dfrac{f(x)}{x}$

$\Rightarrow 0 < \dfrac{f(x)}{x} < 1$, multiply both sides by x. $(x > 0)$ To get

$\boxed{0 < f(x) < x,\ x > 0}$

MAT 101E **MIDTERM** **09.04.2011**

QUESTION 4

Student Number:	e-mail:		Group:	List No.:	Grade
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[25p] Graph the curve $f(x) = \dfrac{3x^2 + 4x + 1}{x^2 - 1}$ by considering its

i) domain,
ii) interception points,
iii) asymptotes if any,
iv) maxima, minima and the intervals that the curve is increasing/decreasing,
v) concavity and inflection points if any.

i) $x^2 - 1 = 0 \Rightarrow$ function is discontinuous at $x = \mp 1$,

Domain: $\mathbb{R} - \{-1, 1\}$.

ii) $x = 0 \Rightarrow y = -1$, $(0, -1)$
$y = 0 \Rightarrow x = -\frac{1}{3}$ $\quad(-\frac{1}{3}, 0)$

iii) $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \dfrac{(3x+1)\cancel{(x+1)}}{(x-1)\cancel{(x+1)}} = \infty$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \dfrac{(3x+1)\cancel{(x+1)}}{(x-1)\cancel{(x+1)}} = -\infty$

} vertical asymptote at $x = 1$

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \dfrac{(3x+1)\cancel{(x+1)}}{(x-1)\cancel{(x+1)}} = 1$

$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \dfrac{(3x+1)\cancel{(x+1)}}{(x-1)\cancel{(x+1)}} = 1$

} No asymptotes at $x = -1$ (There is a removable discontinuity)

$\lim_{x \to \mp\infty} f(x) = 3 \Rightarrow y = 3$ horizontal asymptote

iv) $f'(x) = \dfrac{-4}{(x-1)^2}$

	$-\infty$	1	∞
$f'(x)$	$-$	‖	$-$

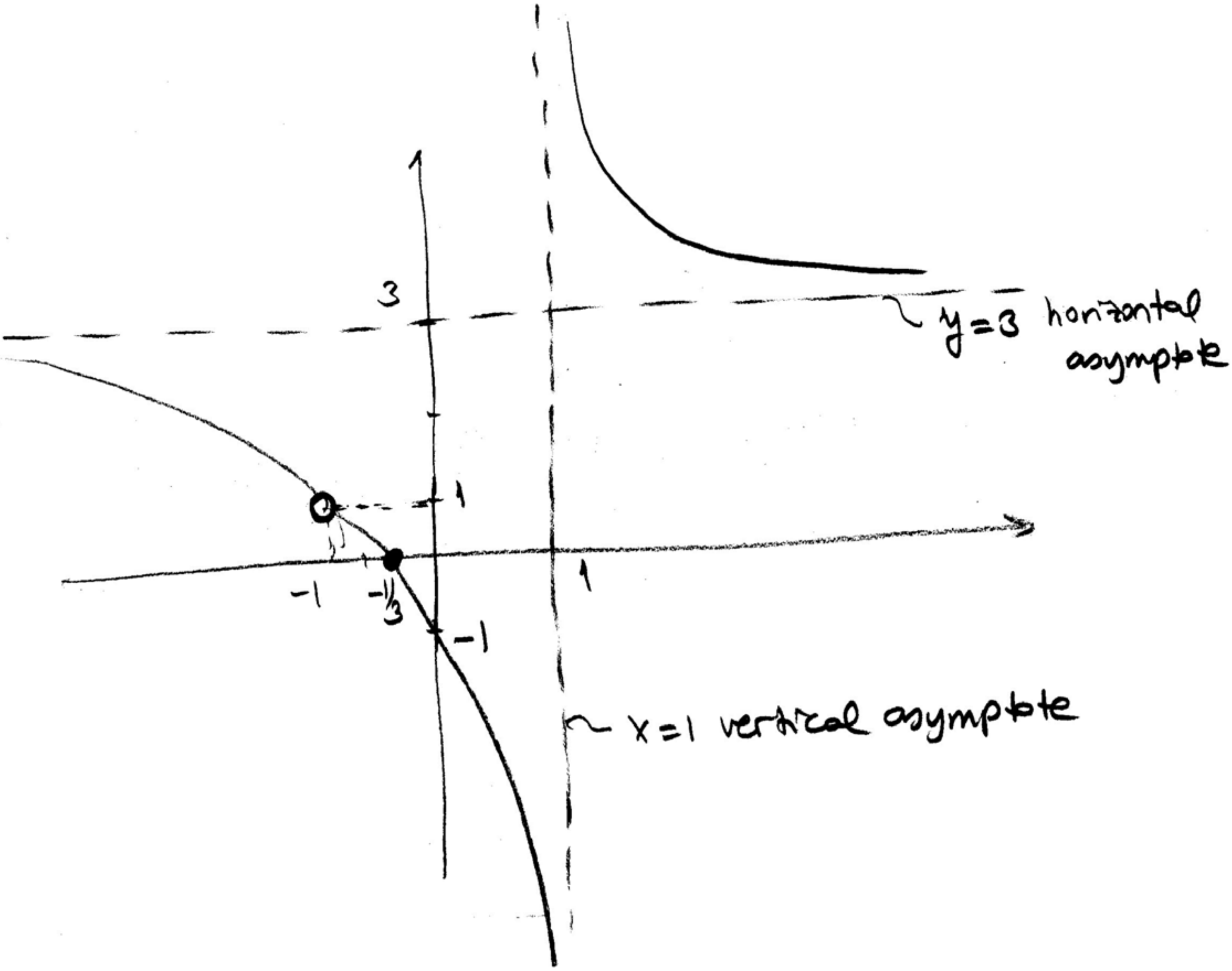
Decreasing

v) $f''(x) = \dfrac{8}{(x-1)^3}$

	$-\infty$	1	∞
$f''(x)$	$-$	‖	$+$

concave down $\quad$ concave up.

No inflection points.


3
1
-1
-1/3
1
-1
y=3 horizontal asymptote
x=1 vertical asymptote