WORKSHEET-3

Course: MAT101E
Subject: Derivatives

1. Using the definition, find the derivatives of the following functions. Then evaluate the derivatives at the specified points.

   (a) $f(x) = (x-1)^2 + 1; \quad f'(-1),\ f'(0),\ f'(2).$

   (b) $f(x) = \frac{1}{\sqrt[3]{x}}; \quad f'(-1),\ f'(2).$

   (c) $f(x) = \cos(2x+1); \quad f'(-1/2),\ f'(\frac{\pi-2}{4}).$

   (d) $f(x) = \frac{1}{\sqrt{x}}; \quad f'(1),\ f'(1/2).$

2. Show that the following functions are differentiable at $x = 0$.

   (a) $f(x) = |x| \sin x$

   (b) $f(x) = x^{2/3} \sin x$

   (c) $f(x) = \sqrt[3]{x}\,(1 - \cos x)$

   (d) $h(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0 \quad\quad , & x = 0 \end{cases}$

3. Let $f(x) = \begin{cases} \sqrt{x} & 0 \leq x \leq 1 \\ 2x-1 & x > 1 \end{cases}$. Is f differentiable at $x = 1$? Give your reasons by using the definition of differentiability.

4. Find the values of a and b that make

$$f(x) = \begin{cases} ax + b & x < 0 \\ 2\sin x + 3\cos x & x \geq 0 \end{cases}$$

   differentiable at $x = 0$?

5. (a) Graph the function $f(x) = \begin{cases} x^3 & x < 1 \\ \sqrt{x} & x \geq 1. \end{cases}$

   (b) Is f continuous at $x = 1$?

   (c) Is f differentiable at $x = 1$?

   Give reasons for your answers.

6. Let $f(x) = \begin{cases} x + b & x < 0 \\ \cos x & x \geq 0 \end{cases}$.

   (a) Is there a value of b that makes $f(x)$ continuous at $x = 0$? If so, what is it? If not, why not?

   (b) Is there a value of b that makes $f(x)$ differentiable at $x = 0$? If so, what is it? If not, why not?

7. Does the curve $y = 2\sqrt{x}$ have a horizontal tangent line? Explain your answer.

8. Does the parabola $y = 2x^2 - 13x + 5$ have a tangent line with slope -1? If so, find the equation of that tangent line.

9. (a) Let $f(x)$ be a function defined on $[-1, 1]$ for which $|f(x)| \leq x^2$ holds. Show that $f(x)$ is differentiable at $x = 0$ and find $f'(0)$.

   (b) Using (a), evaluate $g'(0)$ for the function

$$g(x) = \begin{cases} x^2 sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

10. Find the first and second order derivatives of the following functions.

   (a) $y = (x^2 + 1)(x + 5 + \frac{1}{x})$

   (b) $y = (1 - x)^4(1 + sin^2x)^{-5}$

   (c) $y = \left(\frac{1 - \sqrt{x}}{1 + \sqrt{x}}\right)^4$

   (d) $y = \frac{1}{(x^2 + 1)(x^2 + x + 1)}$

11. If u and v are differentiable functions of x and $u(1) = 2$, $u'(1) = 0$, $v(1) = 5$, $v'(1) = -1$, evaluate the following derivatives at $x = 1$.

(a) $\frac{d}{dx}(uv)$, (b) $\frac{d}{dx}(\frac{u}{v})$, (c) $\frac{d}{dx}(\frac{v}{u})$, (d) $\frac{d}{dx}(7v - 2u)$.

12. Is there a value of the constant m for which the function $f(x) = \begin{cases} \sin 2x, & x \leq 0 \\ mx\ , & x > 0 \end{cases}$

   (a) is continuous at $x = 0$?

   (b) is differentiable at $x = 0$?

   Give reasons for your answers.

13. For which values of a and b is the function

$$f(x) = \begin{cases} ax + b & , \quad x \leq -1 \\ ax^3 + x + 2b, & x > -1 \end{cases}$$

   differentiable for all real numbers?

14. Consider the function

$$f(x) = \begin{cases} 3x, & , \quad x < 0 \\ -(2 - x)^2, & 0 \leq x \leq 2 \\ x^2 - 4 & , \quad x > 2 \end{cases} .$$

   (a) Analyze the continuity of $f(x)$ at $x = 0$ and $x = 2$?

   (b) Analyze the differentiability of $f(x)$ at $x = 0$ and $x = 2$?

Give reasons for your answers.

15. Find the derivatives of the following functions:

(a) $y = x^7 + \sqrt{7}x - \dfrac{1}{\pi + 1}$,

(b) $y = (x+1)^2(x^2+2x)$,

(c) $y = (2x-5)(4-x)^{-1}$,

(d) $y = \left(-1 - \dfrac{\csc\theta}{2} - \dfrac{\theta^2}{4}\right)^2$,

(e) $y = 2\tan^2 x - \sec^2 x$,

(f) $y = x^{-3}\sec(2x)^2$,

(g) $y = 5\cot x^2$.

16. Find the derivatives of the following functions:

(a) $y = \dfrac{t}{\sqrt{t+1}}$,

(b) $y = (\sec x + \tan x)(\sec x - \tan x)$,

(c) $y = \dfrac{\sin t}{1 - \cos t}$,

(d) $y = \dfrac{\tan x}{1 + \tan x}$,

(e) $y = \sin x^3$,

(f) $y = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^4$,

(g) $y = \left(\frac{x}{5} + \frac{1}{5x}\right)^5$,

(h) $y = (x^2 + 3x + 1)^{1/4}$,

(i) $y = \left(\frac{\sin x}{1 + \cos x}\right)^2$,

(j) $y = 4\sin(\sqrt{x})$.

17. Find the derivatives of the following functions:

(a) $y(x) = \dfrac{\sqrt{x}}{1 + \sqrt{x}} + (\sec x + \tan x)^5$,

(b) $y(x) = \sin(x + \sqrt{x+1}\,) + \left(\frac{1 - x^3}{2x^2 + 5}\right)^2$.

18. Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ at which the tangent line is

(a) perpendicular to the line $y = 1 - \frac{x}{24}$,

(b) parallel to the line $y = \sqrt{2} - 12x$.

19. The curve $y = ax^2 + bx + c$ passes through the point $(1, 2)$ and is tangent to the line $y = x$ at the origin. Find the constants a, b, and c.

20. Suppose that f is continuous and it satisfies the following conditions for all real values of x and y:

(a) $f(x+y) = f(x) \cdot f(y)$;

(b) $f(x) = 1 + xg(x)$, where $\lim_{x \to 0} g(x) = 1$.

Show that the derivative $f'(x)$ exists at every value of x and that $f'(x) = f(x)$.

21. Find d^2y/dx^2 of the following functions:

(a) $x^3 + y^3 = 1$, (b) $y^2 = 1 - \frac{2}{x}$.

22. By differentiating $x^2 - y^2 = 1$ implicitly, show that:

(a) $dy/dx = x/y$,

(b) $d^2y/dx^2 = -1/y^3$.

23. Use implicit differentiation to find dy/dx for the following:

(a) $x^2y + xy^2 = 6$

(b) $x + \sin y = xy$

(c) $y^2 \cos \dfrac{1}{y} = 2x + 2y$

(d) $y^2 = \dfrac{x-1}{x+1}$

(e) $2xy + y^2 = x + y$

(f) $x^2(x-y)^2 = x^2 - y^2$

24. Find an equation for the tangent line to each of the following parametrized curves at the given value. Also, find the value of $\dfrac{d^2y}{dx^2}$ at the given point.

    (a) $x = \sec^2 t - 1 \ , \ y = \tan t \ ; \quad t = -\pi/4,$
    (b) $x = -\sqrt{t+1} \ , \ y = \sqrt{3t} \ ; \quad t = 3,$
    (c) $x = 2t^2 + 3 \ , \ y = t^4 \ ; \quad t = -1,$
    (d) $x = t - \sin t \ , \ y = 1 - \cos t \ ; \quad t = \pi/3,$
    (e) $x = t \ , \ y = \sqrt{t} \ ; \quad t = 1/4.$

25. If $y^3 + y = 2\cos x$, find the value of $\dfrac{d^2y}{dx^2}$ at the point $(0, 1)$.

26. Find an equation of the tangent line to the curve $x \sin(xy - y^2) = x^2 - 1$ at $(1, 1)$.

27. Find an equation of the line tangent to the curve $(x^2+1)y + \frac{1}{\pi}\sin(\pi(y + \sqrt{x})) = 2$ at the point $(1, 1)$.

28. Find the points on the curve $x^2 + xy + y^2 = 7$; (a) at which the tangent line is parallel to the x-axis and (b) at which the tangent line is parallel to the y-axis.

29. Assuming that the following equations define a parametrized curve giving x and y implicitly as differentiable functions of t, find the slope of the curve at the given value.

$$x \sin t + \sqrt{x} = t \ , \quad t \sin t - 2t = y \ , \quad t = \pi.$$