WORKSHEET 2

Course: Mat101E / Mat103E
Content: Limit and Continuity

1. Use $\epsilon - \delta$ definition to prove that:

   (a) $\lim_{x\to -4} \sqrt{1-2x} = 3$ (b) $\lim_{x\to\sqrt{3}} \frac{1}{x^2} = \frac{1}{3}$

   (c) $\lim_{x\to 2} \frac{x^2-1}{x+3} = \frac{3}{5}$ (d) $\lim_{x\to 1} \frac{1}{\sqrt{x}+1} = \frac{1}{2}$

   (e) $\lim_{x\to 2} (x-2)^3 \sin\frac{1}{x-2} = 0$ (f) $\lim_{x\to 1} (x^2-1)\cos\frac{1}{x-1} = 0$

   (g) $\lim_{x\to 1} (x^2+x+3) = 5$ (h) $\lim_{x\to 1} (x^2+x+3) \neq 1$

2. For the following limits find the appropriate value of δ that corresponds to the given ϵ value:

   (a) $\lim_{x\to -1} (1-2x) = 3,\ \epsilon = 0.01$ (b) $\lim_{x\to 0} \frac{1}{x-1} = -1,\ \epsilon = 0.5$ (c) $\lim_{x\to 2} \sqrt{11-x} = 3,\ \epsilon = 1$

3. Find the following limits if they exist or explain why they do not exist.

   (a) $\lim_{x\to 64} \frac{\sqrt{x}-8}{\sqrt[3]{x}-4}$

   (b) $\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}$

   (c) $\lim_{x\to 27} \frac{\sqrt{x}-3\sqrt{3}}{\sqrt[3]{x}-3}$

   (d) $\lim_{x\to 0} \frac{\sqrt[3]{x+1}-1}{\sqrt[4]{x+1}-1}$

   (e) $\lim_{x\to 0} \frac{x}{\tan 3x}$

   (f) $\lim_{x\to \pi} \frac{\sin x}{x-\pi}$

   (g) $\lim_{x\to 0} \frac{\sin^2 x}{x(1-\cos x)}$

   (h) $\lim_{x\to \infty} \frac{x+\sin x}{x+\cos x}$

   (i) $\lim_{x\to \mp\infty} \tan^{-1} x$

   (j) $\lim_{x\to \infty} \frac{\sqrt{x}+x}{\sqrt{x}+\cos x}$

   (k) $\lim_{x\to \infty} e^x \sin(e^{-x})$

   (l) $\lim_{x\to \infty} \frac{x^2+2}{x-5}$

   (m) $\lim_{x\to \infty} \frac{x-5}{x^2+2}$

   (n) $\lim_{h\to 0} \frac{\cos h-1}{h}$

   (o) $\lim_{x\to \infty} \frac{2x}{\sin x}$

   (p) $\lim_{x\to \infty} \frac{x+\sin x}{2x+5}$

   (q) $\lim_{x\to 0} \frac{\tan^{-1} x}{x}$

   (r) $\lim_{x\to 0} \frac{\frac{1}{2+x}-\frac{1}{2}}{x}$

   (s) $\lim_{x\to 0} x\sin\frac{1}{x}$

   (t) $\lim_{x\to 0} \frac{\sin^2 3x}{5x^2}$

   (u) $\lim_{x\to 0} \frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3}$

   (v) $\lim_{x\to 0} \frac{\sqrt{1+x^2}-\sqrt{1+x}}{x}$

   (w) $\lim_{x\to 0} \frac{1+\sin x-\cos x}{1-\sin x-\cos x}$

   (x) $\lim_{x\to \infty} x(\sqrt{9x^2+1}-3x)$

   (y) $\lim_{x\to 3} \frac{3-x}{\sqrt{4-x}-\sqrt{\frac{x}{3}}}$

   (z) $\lim_{x\to \infty} (\sqrt{x^2+1}-\sqrt{x^2-1})$

4. Let $\lim_{x\to 1} f(x) = -1$. Evaluate $\lim_{x\to 1} \frac{\sin(1+f(x))}{1-f^2(x)}$.

5. Find the right-hand and left-hand limits of the following functions at the given point.

(a) $y = \dfrac{|\,x-1\,|}{x-1} + x^2,\ (x = 1)$

(b) $y = \dfrac{x}{x^2-1} + x^2,\ (x = \mp 1)$

(c) $y = \dfrac{1}{x^{4/3}} - \dfrac{1}{(x-2)^{1/3}},\ (x = 0, 2)$

(d) $y = \dfrac{2+x}{1+2^{1/x}},\ (x = 0)$

(e) $y = \begin{cases} 1-x^2 & |\,x\,| \leq 1 \\ \dfrac{1}{|\,x\,|} & |\,x\,| > 1 \end{cases},\ (x = -1)$

(f) $y = \dfrac{\sqrt{1-\cos 2x}}{\sqrt{2}x},\ (x = 0)$

(g) $y = \tan^{-1}\dfrac{x}{x-2}\ ,(x = 2)$

(h) $y = \dfrac{1}{3-4^{\frac{1}{x-2}}}\ ,(x = 2)$

6. Suppose that f is an even function of x. Does knowing that $\lim\limits_{x\to 2^-} f(x) = 7$ tell you anything about either $\lim\limits_{x\to -2^-} f(x)$ or $\lim\limits_{x\to -2^+} f(x)$? Give reasons for your answer.

7. Find the asymptotes of the following functions.

(a) $f(x) = \dfrac{x^3}{4-x^2}$

(b) $f(x) = \dfrac{x^3+2x-1}{x^3+2x^2-x-2}$

8. Let $f(x) = \begin{cases} x+3 & -3 \leq x < -1 \\ -1 & x = -1 \\ -x+1 & -1 < x \leq 1 \\ \dfrac{1}{x-1} & 1 < x \leq 2 \\ x & x > 2 \end{cases}$

(a) Graph $f(x)$.

(b) Find the points, if any, at which the function $f(x)$ is discontinuous and classify the types of the discontinuities.

9. Discuss limit, one-sided limits, continuity and one-sided continuity of f and g at each of the points $x = 0, \mp 1$.

$f(x) = \begin{cases} 1 & x \leq -1 \\ -x & -1 < x < 0 \\ 1 & x = 0 \\ -x & 0 < x < 1 \\ 1 & x > 1 \end{cases}$

$g(x) = \begin{cases} 0 & x \leq -1 \\ \dfrac{1}{x} & |\,x\,| < 1 \\ 0 & x = 1 \\ 1 & x > 1 \end{cases}$

10. For the following functions find the discontinuity points, if any, and classify the types of the discontinuities.

(a) $f(x) = \dfrac{x-2}{x+2}$

(b) $f(x) = \dfrac{x^2+1}{x^2-4x+3}$

(c) $f(x) = \dfrac{1}{x^2+1}$

(d) $f(x) = \dfrac{|\,x\,|}{x}$

(e) $f(x) = \dfrac{1}{1-3^{\frac{3-x}{x}}}$

(f) $f(x) = \sqrt{x}\sin\dfrac{1}{x}$

(g) $f(x) = \begin{cases} \sin^{-1}\dfrac{x}{2} & 0 < x < 2 \\ \pi & x = 2 \\ \tan^{-1}\dfrac{1}{x-2} & x > 2 \end{cases}$

(h) $f(x) = \begin{cases} \dfrac{1-\cos x}{x^2} & x \neq 0 \\ 1 & x = 0 \end{cases}$

11. Evaluate the following limit (Do not use the L'Hospital's Rule).

$$\lim_{x \to 1^+} \{\ln[sin(x^2-1)] - \ln(x-1)\}$$

12. Define $f(1)$ in a way that extends $f(x) = \dfrac{x^2+2x-3}{x^2-1}$ to be continuous at $x = 1$.

13. For what value of a is $f(x) = \begin{cases} x^2 - 1 & x < 3 \\ 2ax & x \geq 3 \end{cases}$ continuous at every x?

14. If $x^4 \leq f(x) \leq x^2$ for all x in $[-1, 1]$ and $x^2 \leq f(x) \leq x^4$ for all $x \in (-\infty, \infty)$, at which point(s) c do you automatically know $\lim_{x \to c} f(x)$? What is the value of the limit at these points?

15. Let $f(x) = x^3 - 2x + 2$. Show that f must have a zero between -2 and 0.

16. Show that the following functions have at least one root.
    (a) $f(x) = \sqrt[3]{x} + x - 2$
    (b) $f(x) = \cos x + \sin x - x$

17. Suppose that a function f is continuous on the closed interval $[0, 1]$ and that $0 \leq f(x) \leq 1$ for every $x \in [0, 1]$. Show that there must exist a number $c \in [0, 1]$ such that $f(c) = c$.

18. Suppose that f and g are continuous on $[a, b]$ and that $f(a) < g(a)$ but $f(b) > g(b)$. Prove that $f(c) = g(c)$ for some $c \in [a, b]$.

19. If $F(x) = (x-a)^2(x-b)^2 + x$ where $a, b \in R$. Show that there must exist a number $c \in (a, b)$ such that $F(c) = \dfrac{a+b}{2}$.