WORKSHEET-5

Course Title: Mat101E
Content: Integration

1. Evaluate the following integrals

   (a) $\int \frac{t\sqrt{t}+\sqrt{t}}{t^2}\,dt,$

   (b) $\int \frac{9r^2}{\sqrt{1-r^3}}\,dr,$

   (c) $\int \frac{\sin(2t+1)}{\cos^2(2t+1)}\,dt,$

   (d) $\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)}\,dx,$

   (e) $\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\,\sin^2\sqrt{\theta}}\,d\theta,$

   (f) $\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\,d\theta,$

   (g) $\int (\theta^4-2\theta^2+8\theta-2)(\theta^3-\theta+2)\,d\theta,$

   (h) $\int x^{1/3}\sin(x^{4/3}-8)\,dx,$

   (i) $\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}}\,d\theta,$

   (j) $\int \frac{1-\cos(6t)}{2}\,dt,$

   (k) $\int \frac{x+\sin x}{1+\cos x}\,dx,$

   (l) $\int \frac{\sqrt{\tan x}}{\sin x\cos x}\,dx,$

   (m) $\int \frac{dx}{1+\cos x},$

2. Evaluate the following integrals

   (a) $\int x\sec x^2\tan x^2\,dx,$

   (b) $\int \frac{\cos^2 y}{7}\,dy,$

   (c) $\int \frac{x^4}{\sin^2 x^5}\,dx,$

   (d) $\int 1-\cot^2 x\,dx,$

(e) $\displaystyle\int \frac{\csc\theta}{\csc\theta - \sin\theta}\, d\theta$,

(f) $\displaystyle\int \sin^{3/4} x \cos^3 x\, dx$,

(g) $\displaystyle\int \sin^2 x \cos^2 x\, dx$

(h) $\displaystyle\int \sqrt{1+\cos 3x}\, dx$

(i) $\displaystyle\int \sqrt[3]{\frac{2x^2+3}{x^{11}}}\, dx$

(j) $\displaystyle\int \sqrt{\frac{x^2-2x}{x^6}}\, dx$

(k) $\displaystyle\int \cos(3z+4)\, dz$

(l) $\displaystyle\int y^3(y^4+1)^8\, dy$

(m) $\displaystyle\int \tan^2\theta\, d\theta$

(n) $\displaystyle\int \sin^2 5t \cos 5t\, dt$

(o) $\displaystyle\int \left(1-\cos\frac{x}{2}\right)^2 \sin\frac{x}{2}\, dx$

(p) $\displaystyle\int \frac{18\tan^2 x\, \sec^2 x}{(2+\tan^3 x)^2}\, dx$

(q) $\displaystyle\int \frac{(2x-1)\, \cos\sqrt{3(2x-1)^2+6}}{\sqrt{3(2x-1)^2+6}}\, dx$

3. Verify by differentiation that the integral formulas

$$\int \sin x \cos x\ dx = \frac{1}{2}\sin^2 x + c_1$$

and

$$\int \sin x \cos x\ dx = -\frac{1}{2}\cos^2 x + c_2$$

are both valid. Reconcile these seemingly different results. What is the relation between the constants c_1 and c_2?

4. Evaluate the following integrals by computing $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(x_i)\Delta x$.

(a) $\displaystyle\int_0^2 x^2 dx$ (b) $\displaystyle\int_1^5 (4-3x)dx$

5. Express the following limits as definite integrals.

(a) $\displaystyle\lim_{||p||\to 0}\sum_{k=1}^{n} c_k^2\, \Delta x_k$, where P is a partition of $[0,2]$

(b) $\lim_{||p|| \to 0} \sum_{k=1}^{n} (c_k^2 - 3c_k)\,\Delta x_k$, where P is a partition of $[-7, 5]$

(c) $\lim_{||p|| \to 0} \sum_{k=1}^{n} \frac{1}{1 - c_k}\,\Delta x_k$, where P is a partition of $[2, 3]$

(d) $\lim_{||p|| \to 0} \sum_{k=1}^{n} \sqrt{4 - c_k^2}\,\Delta x_k$, where P is a partition of $[0, 1]$

(e) $\lim_{||p|| \to 0} \sum_{k=1}^{n} \sec c_k\,\Delta x_k$, where P is a partition of $[-\pi/4, 0]$

6. Find the following limits by using the definition of definite integral or sum formulas.

(a) $\lim_{n \to \infty} \frac{1}{n^3} \left(1^2 + 2^2 + 3^2 + \cdots + n^2 \right)$

(b) $\lim_{n \to \infty} \frac{1}{n^4} \left(1^3 + 2^3 + 3^3 + \cdots + n^3 \right)$

(c) $\lim_{n \to \infty} \frac{1}{n} \left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \sin\frac{3\pi}{n} + \ldots + \sin\frac{n\pi}{n} \right)$

(d) $\lim_{n \to \infty} \frac{1}{n^{3/2}} \left(1 + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n} \right)$

7. Find the indicated derivatives for the given functions

(a) $y(x) = \int_0^x \sqrt{1 + t^2}\, dt \;\;,\;\; y' = ?$

(b) $y(x) = \int_{\sqrt{x}}^0 \sin t^2\, dt \;\;,\;\; y' = ?$

(c) $y(x) = \int_{\tan x}^0 \frac{1}{t^2}\, dt \;\;,\;\; y'' = ?$

(d) $y(x) = \int_{\sqrt{x}}^{x^2} t\sqrt{t^2 + 1}\, dt \;\;,\;\; y' = ?$

(e) $y(x) = \int_0^{\sin x} \sqrt{1 - t^2}\, dt \;\;,\;\; y'' = ?$

(f) $y(x) = \int_{\sin x}^{\cos x} \frac{1}{1 - t^2}\, dt \;\;,\;\; y' = ?$

8. If $F(t) = \int_0^t \cos(x^2)\, dx$, find $\frac{d}{dx} F(\sqrt{x})$.

9. If $H(x) = 3x \int_4^{x^2} e^{-\sqrt{t}}\, dt$, find $H'(2)$.

10. Find a function f such that

$$x^2 = 1 + \int_1^x \sqrt{1 + [f(t)]^2}\, dt$$

for all $x > 1$.

11. Suppose that $f(x) = f(x+w)$ for all x and f is continuous.
We define $g(x) = \int_x^{x+w} f(t)\, dt$ for all x.
Prove that the function $g(x)$ is constant.

12. Evaluate the following integrals

   (a) $\int_0^1 (1-2x)^3\, dx$

   (b) $\int_9^4 \frac{1-\sqrt{u}}{\sqrt{u}}\, du$

   (c) $\int_{-4}^4 |x|\, dx$

   (d) $\int_2^{-2} |1-x|\, dx$

   (e) $\int_5^0 (2-|x|)\, dx$

   (f) $\int_6^0 |5-|2x||\, dx$

   (g) $\int_0^{\pi/4} \sin^2(4x - \frac{\pi}{4})\, dx$

13. The fundamental theorem of calculus seems to say that

$$\int_{-1}^1 \frac{dx}{x^2} = -\frac{1}{x}\Big|_{-1}^1 = -2$$

in apparent contradiction to the fact that $1/x^2$ is always positive. What's wrong here?

14. If $f(x) = f(a-x)$, prove that

$$\int_0^a f(x)\, dx = 2\int_0^{a/2} f(x)\, dx \text{ and } 2\int_0^a x\, f(x)\, dx = a\int_0^a f(x)\, dx$$

15. If $f(x) = f(a-x)$, prove that $2\int_0^a x\, f(x)\, dx = a\int_0^a f(x)\, dx$

16. Prove that for all positive numbers a and b $\int_{t=a}^{ab} \frac{1}{t}\, dt = \int_{t=1}^{b} \frac{1}{t}\, dt$ (Hint: $t = au$)

17. Prove that $\int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\, dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\, dt = \text{constant}.$ $(\sin x\, ,\, \cos x \geq 0)$

18. Prove that $\int_{1/2}^2 \frac{1}{x} \sin\left(x - \frac{1}{x}\right) dx = 0$ (Hint: $x = \frac{1}{t}$)

19. Prove that $\int_0^\pi x\, f(\sin x)\, dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\, dx$ (Hint: $x = \pi - t$)

20. Use properties of integrals to establish each of the following inequality without evaluating the integrals involved.

(a) $\frac{\pi}{2} \leq \int_0^{\frac{\pi}{2}} \sqrt{1 + \frac{1}{2}\sin^2 x}\, dx \leq \frac{\pi}{2}\sqrt{\frac{3}{2}}$

(b) $\frac{\pi}{8} \leq \int_0^{\frac{\pi}{4}} \frac{1}{1+\cos^2 x}\, dx \leq \frac{\pi}{6}$

(c) $1 \leq \int_0^1 \sqrt{1+x^2}\, dx \leq \int_0^1 \sqrt{1+x}\, dx$

(d) $\int_0^1 \frac{1}{1+\sqrt{x}}\, dx \leq \int_0^1 \frac{1}{1+x^2}\, dx$

21. Find $f(4)$ if $\int_0^{x^2} f(t)\, dt = x\cos(\pi x) \quad , \quad x > 0.$

22. What values of a and b maximize the value of $\int_a^b (4x^2 - x^4)\, dx$?

23. Find the area of the region R enclosed by the curves $y = \cos x$; $y = \sin x$ and the y axis in the first quadrant.

24. Find the area of the region R enclosed by the curves $y = x^2$; $y = -x^2 + 2$ and the x axis.

25. Find the area of the region R enclosed by the curves $y = \frac{x^2}{2} + 4$; and $y = |x^2 - 4|$.

26. Find the area of the region R between the graphs of $x = 8 - y^2$ and $x = y^2 - 8$.

27. Find the area of the region bounded below by the circle $x^2 + y^2 = a^2$ and above by the line $y = b$, $(-a \leq b \leq a)$.

28. Find the area of the region bounded by the curves $x = y^2$ and $x = 2y^2 - y - 2$ by integrating with respect to y.

29. Find the area of the finite plane region bounded by the curve $y = x^3$ and the tangent line to that curve at the point $(1, 1)$. (Hint: Find the other point at which that tangent line meets the curve.)

30. Find the areas of the regions enclosed by the following curves.

    (a) $y = x^4 - 2x^2 + 1, \quad y = x^2 - 1$
    (b) $x + 1 = (y-1)^2, \quad (y-1)^2 = 1 - x$
    (c) $y = \frac{1}{2}\sec^2 t, \quad y = -4\sin^2 t, \quad t = \mp\frac{\pi}{3}$

31. Find the area of the region R bounded by the curves $x = 3y^2$ and $x = 12y - y^2 - 5$.

32. Find a number $k > 0$ such that the area bounded by the curves $y = x^2$ and $y = k - x^2$ is 72.

33. Find a number $k > 0$ such that the line $y = k$ divides the region between the parabola $y = 100 - x^2$ and the $x-$axis into two regions having equal areas.

34. Let A and B be the points of intersection of the parabola $y = x^2$ and the line $y = x + 2$, and let C be the point on the parabola where the tangent line is parallel to the graph of $y = x + 2$. Show that the area of the parabolic segment cut from the parabola by the line is four-thirds the area of the triangle ABC.