PROJECT # 2

The Laplace equation in a square enclosure $[-1,1]\times[-1,1]$ is given by

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0 \tag{1}$$

with the Drichlet boundary conditions:

$$\begin{aligned}
\psi(x,-1) &= 0 \quad \text{for} \quad -1<x<1 && (2)\\
\psi(x,+1) &= 1 \quad \text{for} \quad -1<x<1 && (3)\\
\psi(-1,y) &= 0 \quad \text{for} \quad -1<y<1 && (4)\\
\psi(+1,y) &= 0 \quad \text{for} \quad -1<y<1 && (5)\\
& && (6)
\end{aligned}$$

The finite element weak form can be written as:

$$\iint_{\Omega^{(e)}} N_i\left[\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right]dxdy=\iint_{\Omega^{(e)}}\frac{\partial}{\partial x}\left[N_i\frac{\partial\psi}{\partial x}\right]+\frac{\partial}{\partial y}\left[N_i\frac{\partial\psi}{\partial y}\right]dxdy-\iint_{\Omega^{(e)}}\frac{\partial N_i}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial N_i}{\partial y}\frac{\partial\psi}{\partial y}dxdy=0 \tag{7}$$

The above form can be reduced to

$$\oint_{\partial\Omega^{(e)}} N_i\left[n_x\frac{\partial\psi}{\partial x}+n_y\frac{\partial\psi}{\partial y}\right]dS-\iint_{\Omega^{(e)}}\frac{\partial N_i}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial N_i}{\partial y}\frac{\partial\psi}{\partial y}dxdy=0 \tag{8}$$

The present Drichlet boundary conditions lead to

$$\boxed{-\iint_{\Omega^{(e)}}\left[\frac{\partial N_i}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial N_i}{\partial y}\frac{\partial\psi}{\partial y}\right]dxdy=0} \tag{9}$$

For the present FEM discretization we use quadrilateral elements such that

$$x = \sum_{j=1}^{4} x_j N_j \tag{10}$$

$$y = \sum_{j=1}^{4} y_j N_j \tag{11}$$

$$\psi = \sum_{j=1}^{4} \psi_j N_j \tag{12}$$

Then the discretized equation becomes

$$-\sum_{j=1}^{4}\psi_j\iint_{\Omega^{(e)}}\left[\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}+\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right]dxdy=0 \tag{13}$$

In here, the local stiffness matrix entries become

$$k_{ij}=\iint_{\Omega^{(e)}}\left[\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}+\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right]dxdy=0 \tag{14}$$

Please code the above Galerkin FEM code to solve the Laplace equation on a uniform Cartesian mesh with 21×21, 41×41 and 81×81 resolutions. Then compare the numerical solutions with the analytical solution. Finally compute the error given by

$$Error=\frac{1}{\sqrt{Np}}\sqrt{\sum_{i=1}^{Np}(u_i-u_{Analytical})^2} \tag{15}$$

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```
Create Mesh Vertices
for i ← 1 to np do
|  x[i],y[i],z[i]
end

Create Mesh Connectivity
for i ← 1 to ne do
|  nec[i,1],nec[i,2],nec[i,3],nec[i,4]
end

Create Global Coefficient Matrix
for i ← 1 to ne do
|  call Stiffness_Matrix(i,x,y,nec,[K])
|  [A] := [A] + [K]
|  {RHS} := 0
end

Impose Drichlet Boundary Conditions
for i ← 1 to np do
|  if Dirichlet boundary condition is valid for i then
|  |  A[i,*] := 0
|  |  A[i,i] := 1
|  |  RHS[i] := ψ_i
|  end
end

Solve Ax=RHS
```

Table 1: The structure of the Galerkin FEM code.