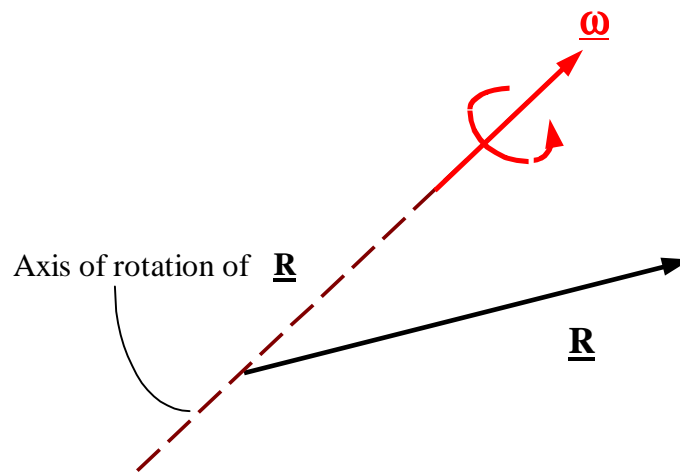


MÜHENDİSLİK MEKANİĞİ

13. HAFTA Kinematics of Rigid bodies

Angular velocity: The angular velocity $\underline{\omega}$ of a vector $\underline{\mathbf{R}}$ is itself a vector which has a magnitude equal to the rate of rotation, and is pointing along the axis of rotation of $\underline{\mathbf{R}}$, following the right-hand rule where the thumb is along the axis of rotation and the other fingers provide the sense in which $\underline{\mathbf{R}}$ is rotating around the axis of rotation. The units of angular velocity are in radians over a suitable unit of time.



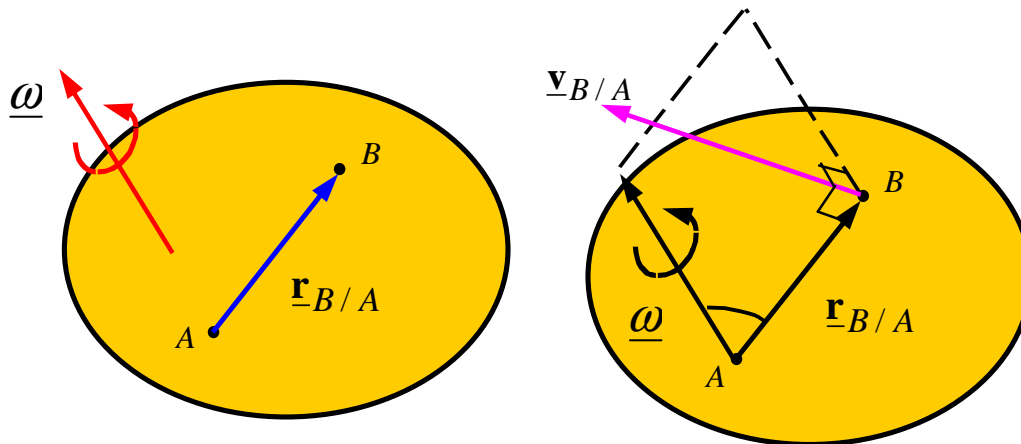
Angular acceleration: The angular acceleration $\underline{\alpha}$ is the rate of change of the angular velocity with respect to time.

$$\underline{\alpha} = \frac{d\underline{\omega}}{dt}$$

The derivative of a constant length vector (lemma): If the length of vector $\underline{\mathbf{R}}$ is constant ($|\underline{\mathbf{R}}| = \text{constant}$), then one can calculate its derivative with respect to time using the formula

$$\dot{\underline{\mathbf{R}}} = \underline{\omega} \times \underline{\mathbf{R}}$$

Relative motion of points on a rigid body



Relative velocity of points on a rigid body: Consider points A and B on a rigid body. The relative position of B with respect to A is given by $\underline{\mathbf{r}}_{B/A}$.

The relative velocity of B with respect to A is given by $\underline{\mathbf{v}}_{B/A} = \dot{\underline{\mathbf{r}}}_{B/A}$. Since the distance between points A and B does not change in a rigid body, $|\underline{\mathbf{r}}_{B/A}|$ is constant and one can calculate the derivative of $\underline{\mathbf{r}}_{B/A}$ using the lemma to get

$$\underline{\mathbf{v}}_{B/A} = \dot{\underline{\mathbf{r}}}_{B/A} = \underline{\boldsymbol{\omega}} \times \underline{\mathbf{r}}_{B/A}$$

where $\underline{\boldsymbol{\omega}}$ is the angular velocity of the rigid body. Since $\underline{\mathbf{v}}_{B/A}$ is the cross product of $\underline{\boldsymbol{\omega}}$ and $\underline{\mathbf{r}}_{B/A}$, the relative velocity of B with respect to A is perpendicular to both $\underline{\boldsymbol{\omega}}$ and $\underline{\mathbf{r}}_{B/A}$.

Relative acceleration of points on a rigid body: The relative acceleration of point B with respect to A is given by the derivative of $\underline{\mathbf{v}}_{B/A}$. Using the above expression for relative velocity, one gets

$$\underline{\mathbf{a}}_{B/A} = \dot{\underline{\mathbf{v}}}_{B/A} = \frac{d}{dt}(\underline{\boldsymbol{\omega}} \times \underline{\mathbf{r}}_{B/A}) = \dot{\underline{\boldsymbol{\omega}}} \times \underline{\mathbf{r}}_{B/A} + \underline{\boldsymbol{\omega}} \times \dot{\underline{\mathbf{r}}}_{B/A} \Rightarrow$$

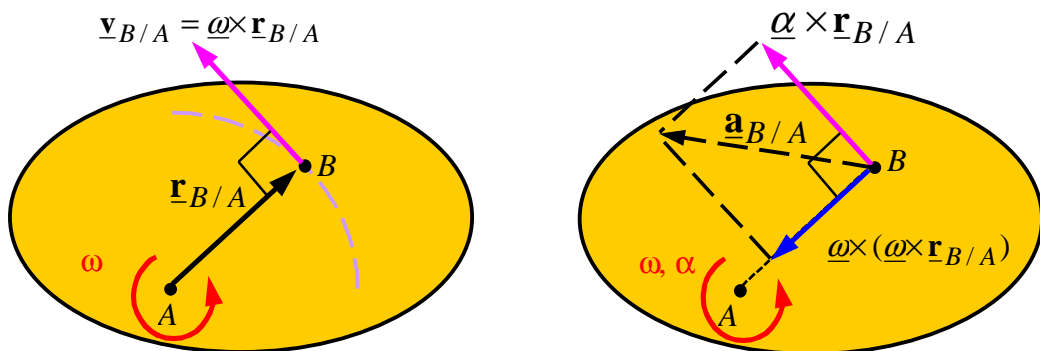
$$\underline{\mathbf{a}}_{B/A} = \underline{\boldsymbol{\alpha}} \times \underline{\mathbf{r}}_{B/A} + \underline{\boldsymbol{\omega}} \times (\underline{\boldsymbol{\omega}} \times \underline{\mathbf{r}}_{B/A})$$

2-D motion: In 2-D, the angular velocity and angular acceleration can be written as

$$\underline{\boldsymbol{\omega}} = \omega \underline{\mathbf{k}}, \quad \underline{\boldsymbol{\alpha}} = \alpha \underline{\mathbf{k}}$$

Example 1:

The following are graphical representations of the relative velocity and relative acceleration of points on a rigid body.

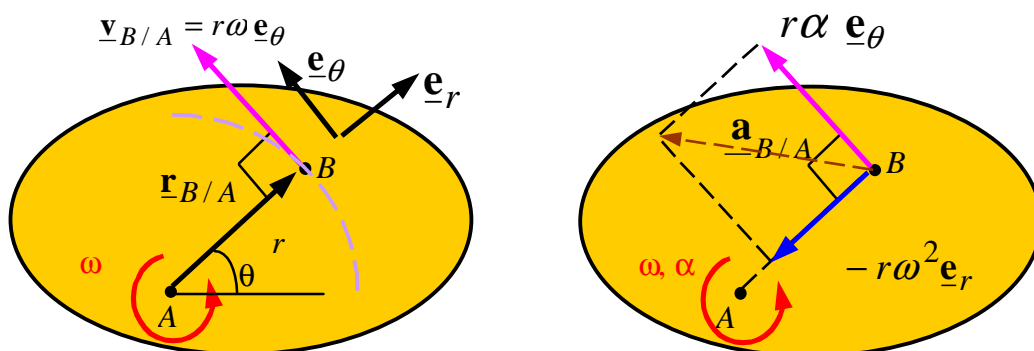


The left figure shows that the motion of point B relative to A describes a circle since the distance between the points does not change. As a result, the velocity of B relative to A is tangent to this circle. The right figure shows the components of acceleration. Since the relative motion is on a circle, the acceleration of B relative to A has a component tangent to the circle ($\underline{\alpha} \times \underline{r}_{B/A}$ = tangential component) and a component towards the center of rotation ($\underline{\omega} \times (\underline{\omega} \times \underline{r}_{B/A})$ = centripetal component).

Since the relative motion of two points on a rigid body always is described by motion on a circular path, one can also use polar coordinates with a constant radial coordinate r (i.e., $\dot{r} = \ddot{r} = 0$), $\dot{\theta} = \omega$, and $\ddot{\theta} = \alpha$. This results in expressions for relative velocity and acceleration given as

$$\underline{v}_{B/A} = r\omega \underline{e}_\theta$$

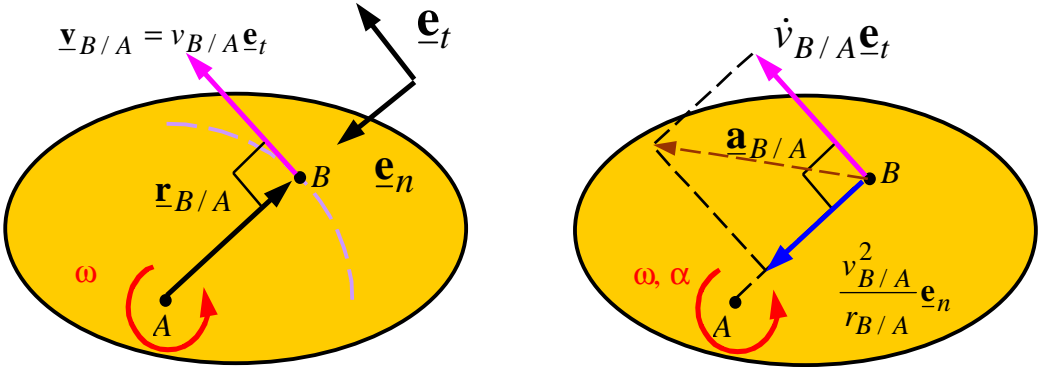
$$\underline{a}_{B/A} = -r\omega^2 \underline{e}_r + r\alpha \underline{e}_\theta$$



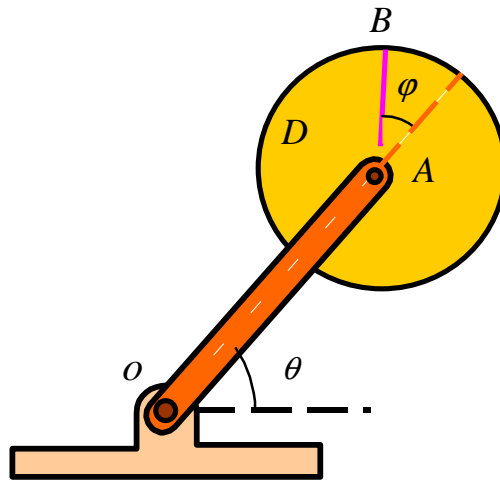
Another method of describing relative motion is by using normal and tangential coordinates. This results in

$$\underline{v}_{B/A} = v_{B/A} \underline{e}_t$$

$$\underline{a}_{B/A} = \dot{v}_{B/A} \underline{e}_t + \frac{v_{B/A}^2}{r_{B/A}} \underline{e}_n$$

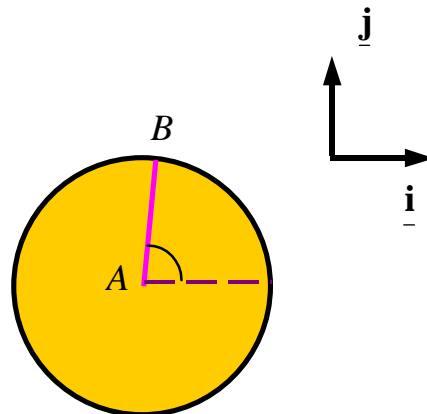


EXAMPLE 1: KINEMATICS OF RIGID BODIES



The bar OA rotates such that $\theta = \theta_o + A \sin(\omega_o t)$, where θ_o , A and ω_o are constants. The disk D rotates relative to bar OA such that $\varphi = \Omega_o t$, where Ω_o is a constant. Calculate the angular velocity and acceleration of D .

SOLUTION:



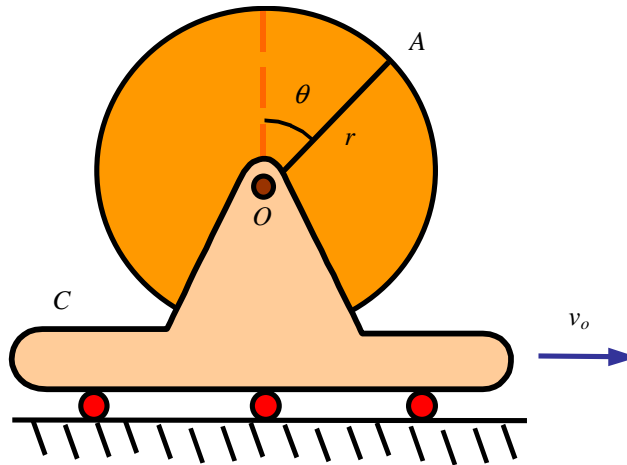
The angle that line AB on D makes with the horizontal is $\varphi + \theta$. The angular velocity of the disk is, therefore,

$$\begin{aligned} \underline{\omega} &= \frac{d}{dt}(\theta + \varphi)\underline{\mathbf{k}} \\ &= [A\omega_o \cos(\omega_o t) + \Omega_o]\underline{\mathbf{k}} \end{aligned}$$

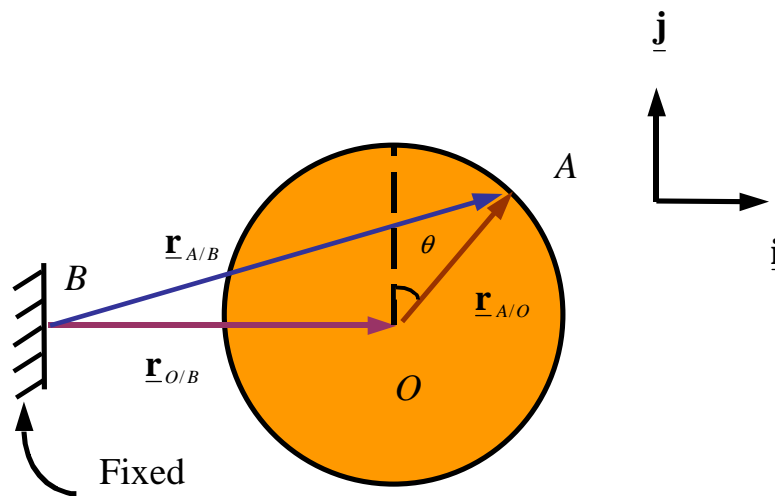
The angular acceleration is

$$\begin{aligned} \underline{\alpha} &= \frac{d}{dt}(\underline{\omega}) = \frac{d}{dt}[A\omega_o \cos(\omega_o t) + \Omega_o]\underline{\mathbf{k}} \\ &= -A\omega_o^2 \sin(\omega_o t)\underline{\mathbf{k}} \end{aligned}$$

EXAMPLE 2: KINEMATICS OF RIGID BODIES



Calculate the velocity of point A given that the cart is moving with a constant speed v_o as shown and the disc is rotating such that $\theta = \omega_o t$, where ω_o is a constant.



SOLUTION:

Position of A relative to fixed point B is

$$\begin{aligned} \underline{r}_{A/B} &= \underline{r}_{A/O} + \underline{r}_{O/B} \\ \Rightarrow \dot{\underline{r}}_{A/B} &= \dot{\underline{r}}_{A/O} + \dot{\underline{r}}_{O/B} \quad (1) \end{aligned}$$

Point O moves with the cart, therefore

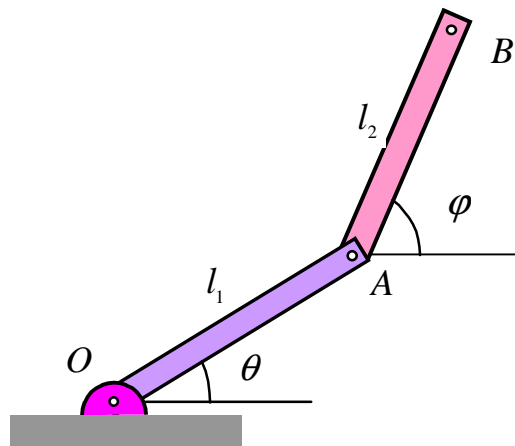
$$\dot{\underline{r}}_{O/B} = \underline{v}_{O/B} = v_o \underline{i}$$

The length of $\underline{\mathbf{r}}_{A/O}$ is constant, therefore

$$\begin{aligned}\dot{\underline{\mathbf{r}}}_{A/O} &= \underline{\boldsymbol{\omega}} \times \underline{\mathbf{r}}_{A/O} \\ &= (-\omega_o \underline{\mathbf{k}}) \times (r \sin \theta \underline{\mathbf{i}} + r \cos \theta \underline{\mathbf{j}}) \\ &= -r \omega_o \sin(\omega_o t) \underline{\mathbf{j}} + r \omega_o \cos(\omega_o t) \underline{\mathbf{i}}\end{aligned}$$

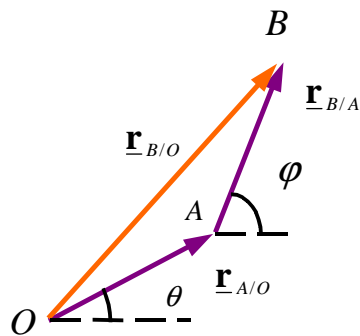
Since point B is stationary, $\dot{\underline{\mathbf{r}}}_{A/B}$ is the velocity of B . Substitution into (1) gives

$$\underline{\mathbf{v}}_{A/B} = (r \omega_o \cos(\omega_o t) + v_o) \underline{\mathbf{i}} - r \omega_o \sin(\omega_o t) \underline{\mathbf{j}}$$



EXAMPLE 3: KINEMATICS OF RIGID BODIES

The robot arm is constructed from two bars OA and AB as shown in the figure. At the instant shown, bar OA is rotating clockwise at an angular speed of ω_1 and bar AB is rotating counter clockwise at a rate of ω_2 . Calculate the velocity of B .



SOLUTION:

$$\begin{aligned}\underline{v}_B &= \frac{d}{dt}(\underline{r}_{B/O}) = \frac{d}{dt}(\underline{r}_{B/A} + \underline{r}_{A/O}) \\ &= \frac{d}{dt}(\underline{r}_{B/A}) + \frac{d}{dt}(\underline{r}_{A/O})\end{aligned}\quad (1)$$

Both OA and AB are of constant length, therefore

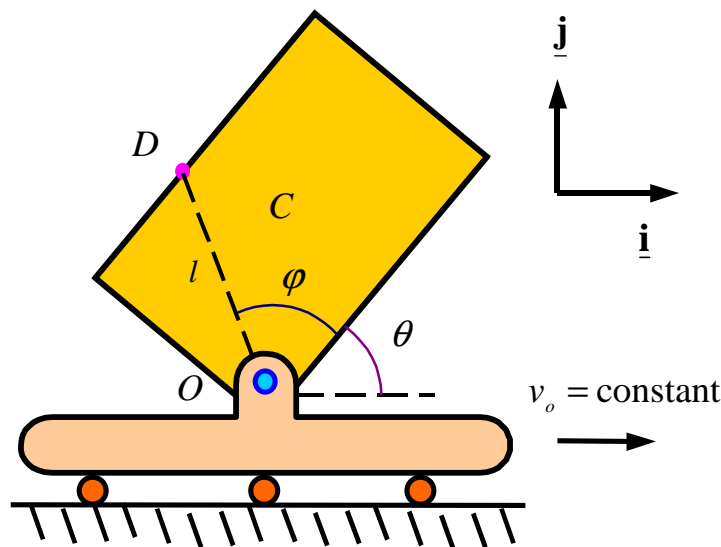
$$\begin{aligned}\frac{d}{dt}(\underline{r}_{A/O}) &= \underline{\omega}_{OA} \times \underline{r}_{A/O} \\ &= (-\omega_1 \underline{k}) \times (l_1 \cos \theta \underline{i} + l_1 \sin \theta \underline{j}) \\ &= -\omega_1 l_1 \cos \theta \underline{j} + \omega_1 l_1 \sin \theta \underline{i}\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}(\underline{r}_{B/A}) &= \underline{\omega}_{AB} \times \underline{r}_{B/A} \\ &= (\omega_2 \underline{k}) \times (l_2 \cos \varphi \underline{i} + l_2 \sin \varphi \underline{j}) \\ &= +\omega_2 l_2 \cos \varphi \underline{j} - \omega_2 l_2 \sin \varphi \underline{i}\end{aligned}$$

Substitution into (1) gives

$$\underline{v}_B = (-\omega_2 l_2 \sin \varphi + \omega_1 l_1 \sin \theta) \underline{i} + (\omega_2 l_2 \cos \varphi - \omega_1 l_1 \cos \theta) \underline{j}$$

EXAMPLE 4: KINEMATICS OF RIGID BODIES



Calculate the velocity and acceleration of point D if at the current instant the cart has a constant velocity of v_o and the rectangular plate C has an angular velocity $\underline{\omega} = \omega \underline{k}$ and angular acceleration $\underline{\alpha} = \alpha \underline{k}$.

SOLUTION:

Velocity of D :

$$\underline{\mathbf{v}}_D = \underline{\mathbf{v}}_O + \underline{\mathbf{v}}_{D/O} \quad (1)$$

Point O moves with the cart

$$\underline{\mathbf{v}}_O = v_o \underline{\mathbf{i}} \quad (2)$$

The length of OD is constant. Therefore,

$$\begin{aligned} \underline{\mathbf{v}}_{D/O} &= \frac{d}{dt}(\underline{\mathbf{r}}_{D/O}) = \underline{\boldsymbol{\omega}} \times \underline{\mathbf{r}}_{D/O} \quad (3) \\ &= (\boldsymbol{\omega} \underline{\mathbf{k}}) \times (l \cos(\theta + \varphi) \underline{\mathbf{i}} + l \sin(\theta + \varphi) \underline{\mathbf{j}}) \\ &= \omega l \cos(\theta + \varphi) \underline{\mathbf{j}} - \omega l \sin(\theta + \varphi) \underline{\mathbf{i}} \end{aligned}$$

Substitution into (1) gives

$$\underline{\mathbf{v}}_D = [v_o - \omega l \sin(\theta + \varphi)] \underline{\mathbf{i}} + \omega l \cos(\theta + \varphi) \underline{\mathbf{j}}$$

Acceleration of D :

$$\underline{\mathbf{a}}_D = \underline{\mathbf{a}}_O + \underline{\mathbf{a}}_{D/O} \quad (4)$$

The cart has constant velocity so that its acceleration is

$$\underline{\mathbf{a}}_O = 0 \quad (5)$$

Using (3) we get

$$\begin{aligned} \underline{\mathbf{a}}_{D/O} &= \frac{d}{dt}(\underline{\mathbf{v}}_{D/O}) = \frac{d}{dt}(\underline{\boldsymbol{\omega}} \times \underline{\mathbf{r}}_{D/O}) \\ &= \frac{d}{dt}(\underline{\boldsymbol{\omega}}) \times \underline{\mathbf{r}}_{D/O} + \underline{\boldsymbol{\omega}} \times \frac{d}{dt}(\underline{\mathbf{r}}_{D/O}) \\ &= \underline{\boldsymbol{\alpha}} \times \underline{\mathbf{r}}_{D/O} + \underline{\boldsymbol{\omega}} \times \underline{\mathbf{v}}_{D/O} \end{aligned}$$

Substitution yields,

$$\begin{aligned} \underline{\mathbf{a}}_{D/O} &= (\alpha \underline{\mathbf{k}}) \times [l \cos(\theta + \varphi) \underline{\mathbf{i}} + l \sin(\theta + \varphi) \underline{\mathbf{j}}] \\ &+ (\boldsymbol{\omega} \underline{\mathbf{k}}) \times [\omega l \cos(\theta + \varphi) \underline{\mathbf{j}} - \omega l \sin(\theta + \varphi) \underline{\mathbf{i}}] \\ &= \alpha l \cos(\theta + \varphi) \underline{\mathbf{j}} - \alpha l \sin(\theta + \varphi) \underline{\mathbf{i}} \\ &- \omega^2 l \cos(\theta + \varphi) \underline{\mathbf{i}} - \omega^2 l \sin(\theta + \varphi) \underline{\mathbf{j}} \quad (6) \end{aligned}$$

Substitution into (4) yields

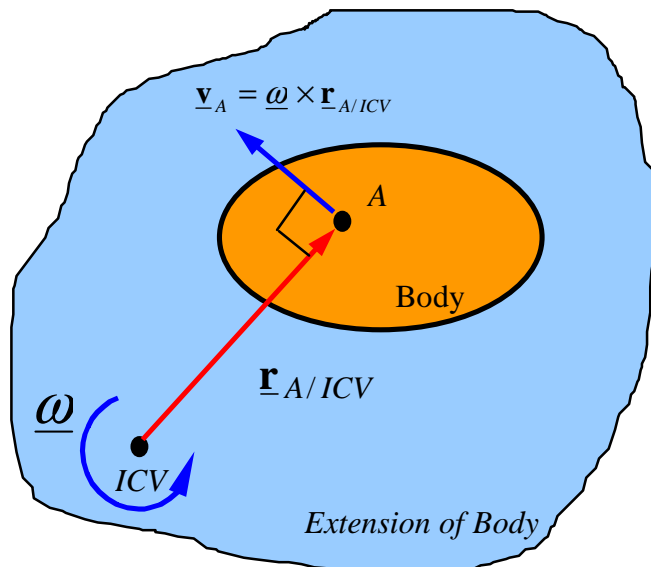
$$\begin{aligned} \underline{\mathbf{a}}_D &= -l[\omega^2 \cos(\theta + \varphi) + \alpha \sin(\theta + \varphi)] \underline{\mathbf{i}} \\ &+ l[\alpha \cos(\theta + \varphi) - \omega^2 \sin(\theta + \varphi)] \underline{\mathbf{j}} \end{aligned}$$

Instantaneous Center of Velocity

Extension of a rigid body: The extension of a rigid body refers to the operation of theoretically extending the body to fill all space. By this operation every point in space becomes a point of the body and as a result has a velocity associated with it. Since this is not an actual extension of the body, a theoretical extension does not influence how the actual body moves-it simply follows the motions of the actual body.

Instantaneous Center of Velocity (ICV): Any point on a rigid body or on its extension that has zero velocity is called the Instantaneous Center of Velocity of the body. Assuming one knows the ICV of a body, one can calculate the velocity of any point A on the body using the equation $\underline{v}_A = \underline{v}_{A/ICV} + \underline{v}_{ICV}$ and recognizing that by definition $\underline{v}_{ICV} = 0$. This gives

$$\underline{v}_A = \underline{\omega} \times \underline{r}_{A/ICV}$$

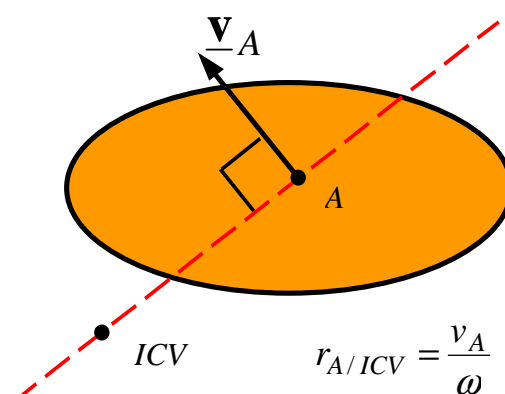


In 2-D motion, if $\underline{r}_{A/ICV}$ is in the plane of motion and $\underline{\omega}$ is perpendicular to this plane, then one can use the scalar relation

$$v_A = \omega r_{A/ICV}$$

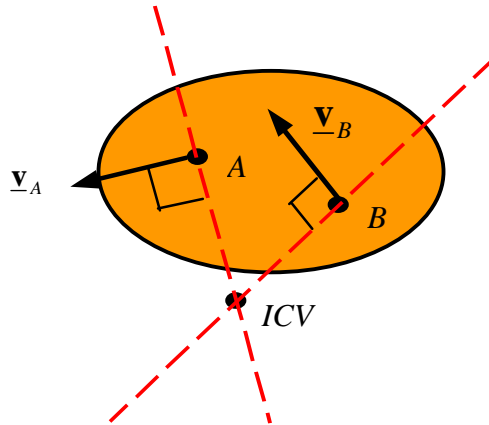
Methods of finding the ICV:

- • Given the velocity \underline{v}_A of point A on a rigid body and the angular velocity of the rigid body one can use the above equation to find the distance

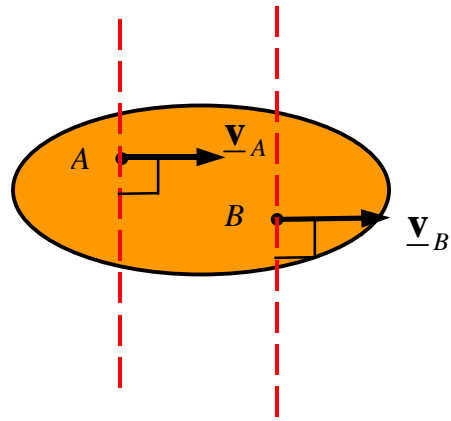


$r_{A/ICV}$ between the point A and the ICV. One can then draw a line perpendicular to the velocity and passing through A , and move along this line a distance $r_{A/ICV}$ to get to the ICV. The side on which the ICV is can be determined by the direction of the angular velocity.

- • Given the velocity of points A and B on a rigid body one can find the ICV by drawing a line perpendicular to \underline{v}_A and passing through A , and by drawing a line perpendicular to \underline{v}_B and passing through B . One of the following three cases will result
 - • **The lines intersect at one point:** The point of intersection is the ICV. The angular velocity can be calculated once the ICV is determined using the velocity of either point and its corresponding distance from the ICV.

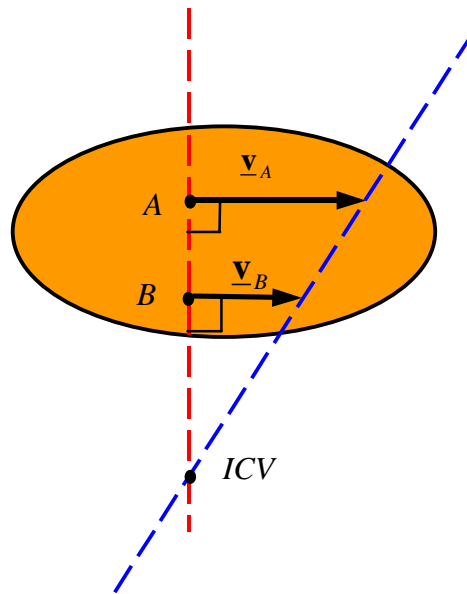


- • **The lines are parallel (they intersect at infinity):** The ICV is at infinity, and the angular velocity is zero since infinity times zero is the only way one can get velocities other than infinite. Therefore, the body is in pure translation and the velocity of the two points must be the same.



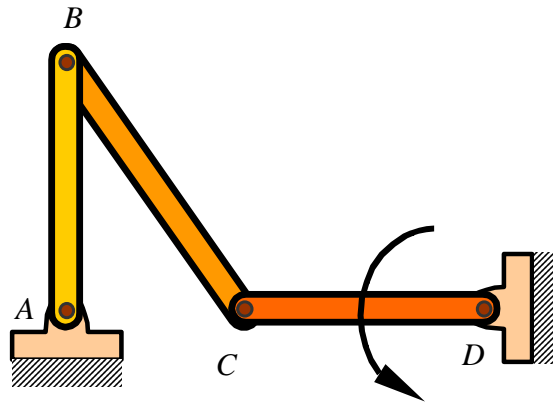
$$ICV \text{ at } \infty \updownarrow \Rightarrow \omega = 0 \Rightarrow \underline{v}_A = \underline{v}_B$$

- • **The two lines fall on top of each other:** One can find the location of the ICV using the proportionality of velocity and distance from the ICV to create similar triangles. This follows from



$$\omega = \frac{v_A}{r_{A/ICV}} = \frac{v_B}{r_{B/ICV}}$$

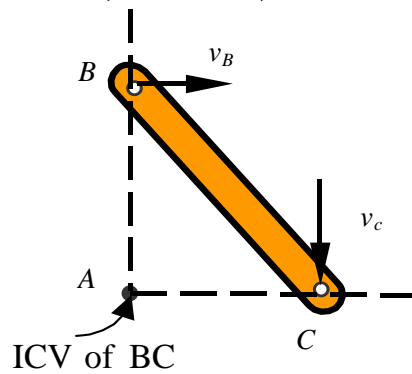
EXAMPLE 1: INSTANTANEOUS CENTER OF VELOCITY



Find the *ICV* of *BC*.

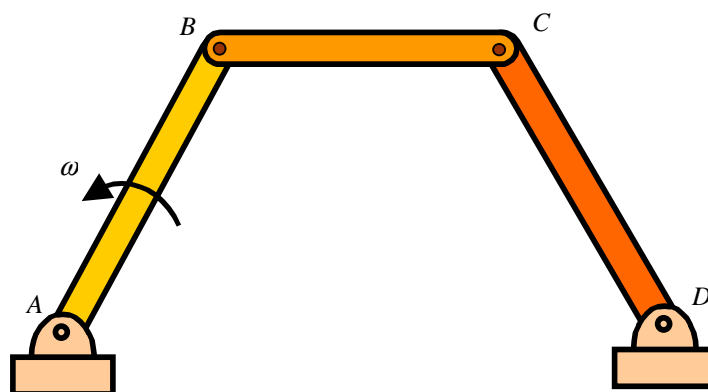
SOLUTION:

B is rotating around *A* and, therefore, has a horizontal velocity.



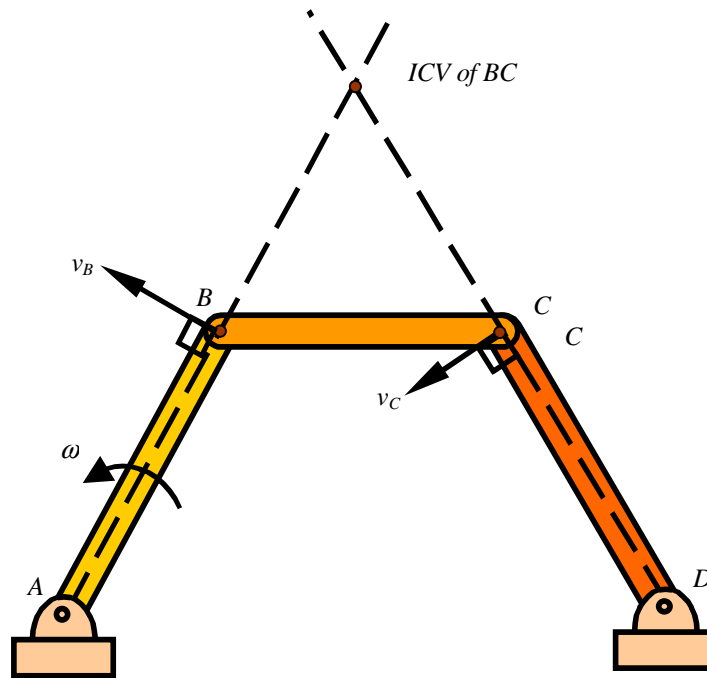
C is rotating in a circle around *D*, and, therefore, has a vertical velocity. The *ICV* is at the intersection of the two perpendicular lines to the velocities.

EXAMPLE 2: INSTANTANEOUS CENTER OF VELOCITY

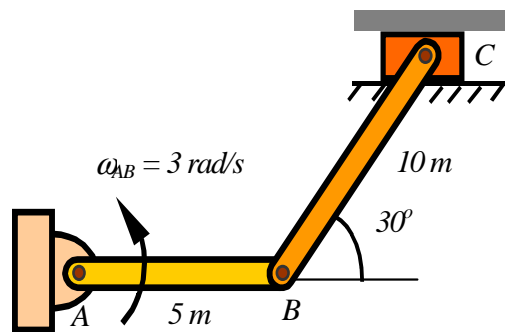


Find the *ICV* of *BC*.

SOLUTION:



EXAMPLE 3: INSTANTANEOUS CENTER OF VELOCITY

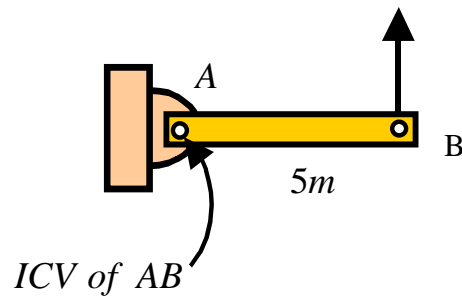


Given that bar AB rotates with an angular velocity of $\omega_{AB} = 3 \text{ rad/s}$, calculate the angular velocity of BC and the velocity of C.

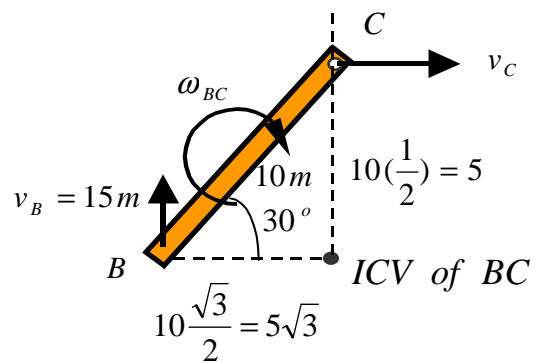
SOLUTION:

Bar AB:

$$v_B = \omega_{AB} r_{B/ICV} = (3)(5) = 15 \text{ m/s}$$



Bar BC:



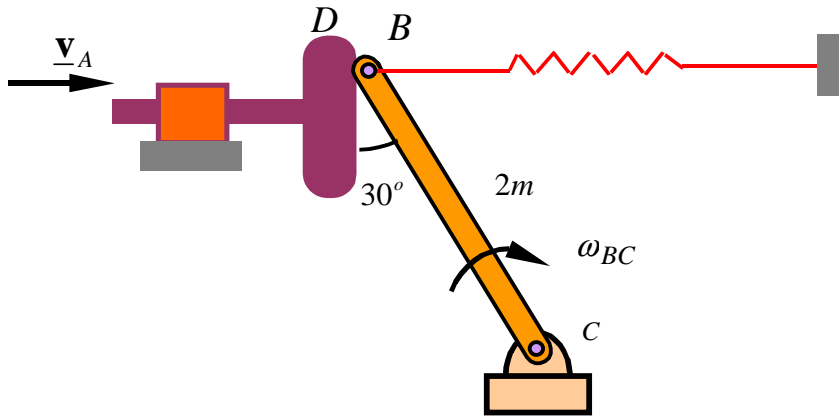
$$v_B = \omega_{BC} r_{B/ICV}$$

$$15 = \omega_{BC} (5\sqrt{3}) \Rightarrow \omega_{BC} = \frac{3}{\sqrt{3}} = \sqrt{3} \text{ rad/s}$$

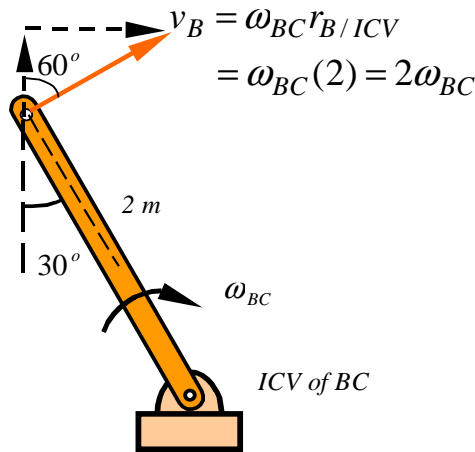
$$v_C = \omega_{BC} r_{C/ICV} = (\sqrt{3})(5) = 5\sqrt{3} \text{ m/s}$$

EXAMPLE 3: INSTANTANEOUS CENTER OF VELOCITY

Given that $v_A = 10 \text{ m/s}$, calculate the angular velocity of BC . Assume that B is in contact with D at all times.



SOLUTION:



First find velocity of B Using the bar BC.

The non-penetration condition requires that the horizontal component of the velocity of B be equal to the horizontal component of velocity of D, therefore,

$$v_B \sin 60 = v_A \Rightarrow 2\omega_{BC} \sin 60 = 10$$

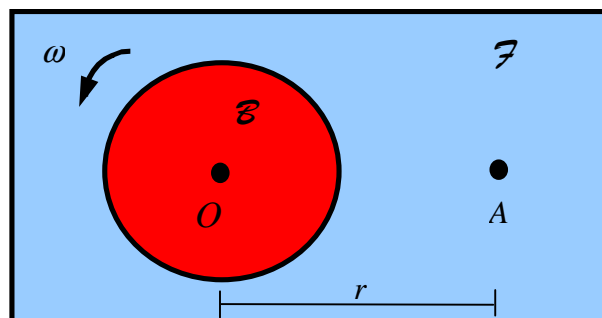
$$\omega_{BC} = \frac{10}{2 \frac{\sqrt{3}}{2}} = \frac{10\sqrt{3}}{3} \text{ rad/s}$$

Motion relative to moving frames

Two observers may provide different description of the same event if they are positioned in different frames. For example, a stationary point on the ground may look to an observer stationed on a merry-go-round as moving. Or, for example, passengers sitting in an airplane may consider each other as stationary, while an observer on earth may consider them to be moving.

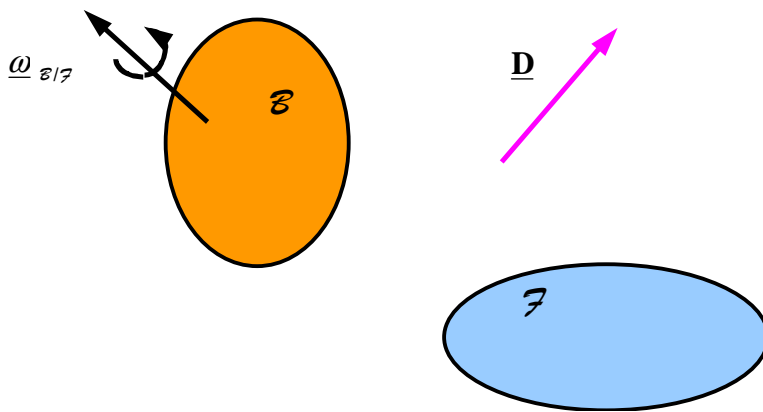
Frames: To describe position in space we use a frame. The frame we use is frequently the background of the event we are observing, yet we can select any rigid body as our frame to reference our observations. The earth is the most common frame we use, but it is not the only one. For example, observations in the shuttle may be done relative to the “shuttle frame,” and then translated to earth frame.

Time derivatives and frames: The derivative of any variable with respect to time provides information on how it is changing with time. Since the same event when viewed from different frames may result in different observations, the time derivative of an event when taken relative to one frame may be different from the time derivative taken relative to another frame. For example, point A which is fixed to frame \mathcal{F} has zero velocity as seen by an observer on frame \mathcal{F} , yet will have a downward velocity of $r\omega$ if observed by an observer sitting at point O on the frame \mathcal{B} , which is a disk rotating about O with angular velocity ω relative to \mathcal{F} .



One can relate the time derivative obtained from an observer in one frame to that obtained from an observer in another frame. Let the vector \mathbf{D} be represent a quantity being observed by two observers, one on frame \mathcal{F} and one on frame \mathcal{B} . Let $[\dot{\mathbf{D}}]_{\mathcal{F}}$ denote the derivative of \mathbf{D} obtained by the observer on \mathcal{F} , and let $[\dot{\mathbf{D}}]_{\mathcal{B}}$ denote the derivative of \mathbf{D} obtained by the observer on \mathcal{B} . These derivatives are related by following equation

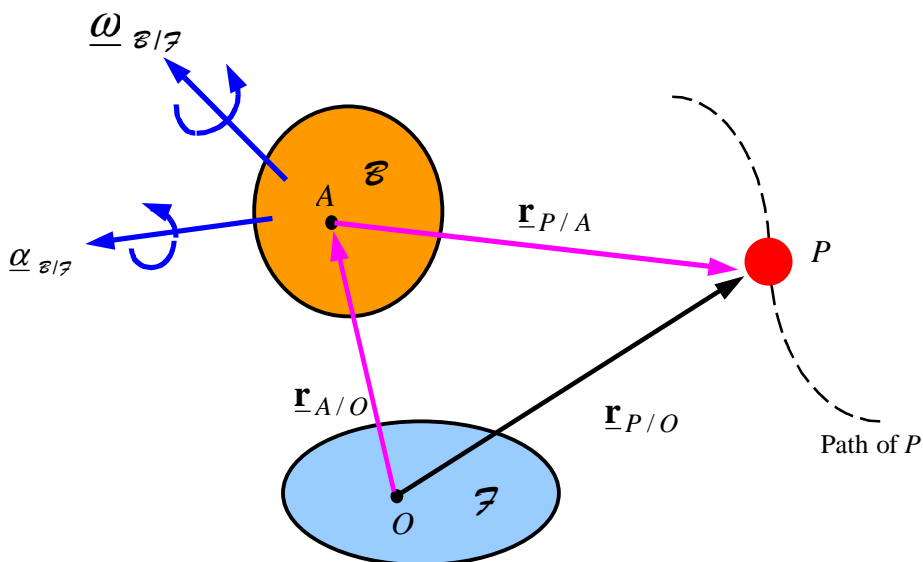
$$[\dot{\mathbf{D}}]_{\mathcal{F}} = [\dot{\mathbf{D}}]_{\mathcal{B}} + \omega_{\mathcal{B}|\mathcal{F}} \times \mathbf{D}$$



where $\underline{\omega}_{B/F}$ is the angular velocity of frame \mathcal{B} with respect to frame \mathcal{F} .

Relating observations on velocity and acceleration: The position of a point P as it moves on its path can be described relative to the origin O of frame \mathcal{F} by the position vector $\underline{r}_{P/O}$ or relative to the origin A of frame \mathcal{B} by the position vector $\underline{r}_{P/A}$. These vectors are related through the law of vector addition given by

$$\underline{r}_{P/O} = \underline{r}_{P/A} + \underline{r}_{A/O}$$



Taking derivatives of both sides of the equation from the point of view of the observer in \mathcal{F} and application of the above relation to relate derivatives in two different frames results in the relation

$$[\underline{\mathbf{v}}_{P/O}]_{\mathcal{F}} = [\underline{\mathbf{v}}_{P/A}]_{\mathcal{B}} + \underline{\omega}_{\mathcal{B}/\mathcal{F}} \times \underline{\mathbf{r}}_{P/A} + [\underline{\mathbf{v}}_{A/O}]_{\mathcal{F}}$$

The first term on the right hand side is the velocity as seen by an observer sitting at point A on frame \mathcal{B} , the second term on the right hand side is a modification of this velocity to account for the rotation of frame \mathcal{B} with respect to \mathcal{F} , and the third term on the right hand side is a modification of the first term to account for the translation of \mathcal{B} with respect to \mathcal{F} . The derivative of this gives the relation between the acceleration as observed from the two frames. After application of the rule for taking derivatives and some reorganization one gets

$$\begin{aligned} [\underline{\mathbf{a}}_{P/O}]_{\mathcal{F}} &= [\underline{\mathbf{a}}_{P/A}]_{\mathcal{B}} + 2\underline{\omega}_{\mathcal{B}/\mathcal{F}} \times [\underline{\mathbf{v}}_{P/A}]_{\mathcal{B}} \\ &+ \underline{\alpha}_{\mathcal{B}/\mathcal{F}} \times \underline{\mathbf{r}}_{P/A} + \underline{\omega}_{\mathcal{B}/\mathcal{F}} \times (\underline{\omega}_{\mathcal{B}/\mathcal{F}} \times \underline{\mathbf{r}}_{P/A}) + [\underline{\mathbf{a}}_{A/O}]_{\mathcal{F}} \end{aligned}$$

The first term on the right hand side is the acceleration measured by an observer located at point A on frame \mathcal{B} , the next three terms are modifications to correct for the rotation of frame \mathcal{B} with respect to \mathcal{F} , and the last term is to correct for the translation of \mathcal{B} with respect to \mathcal{F} .

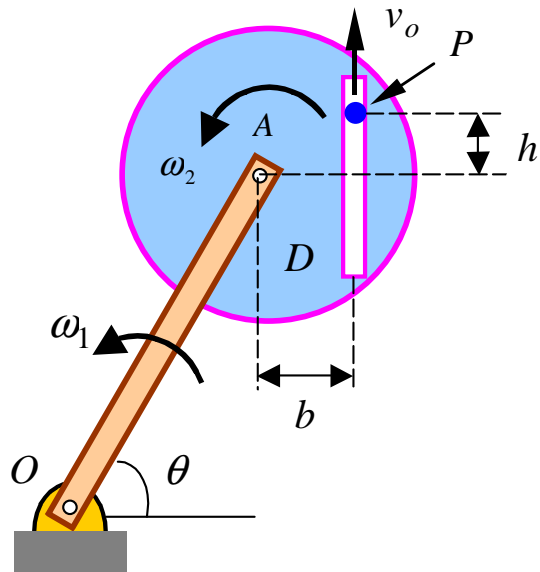
Example

Note: The derivative of $\underline{\omega}_{\mathcal{B}/\mathcal{F}}$ is the same in both frames. This follows from

$$\underline{\alpha}_{\mathcal{B}/\mathcal{F}} = [\dot{\underline{\omega}}_{\mathcal{B}/\mathcal{F}}]_{\mathcal{F}} = [\dot{\underline{\omega}}_{\mathcal{B}/\mathcal{F}}]_{\mathcal{B}} + \underline{\omega}_{\mathcal{B}/\mathcal{F}} \times \underline{\omega}_{\mathcal{B}/\mathcal{F}} = [\dot{\underline{\omega}}_{\mathcal{B}/\mathcal{F}}]_{\mathcal{B}}$$

EXAMPLE 1: MOTION RELATIVE TO MOVING FRAME

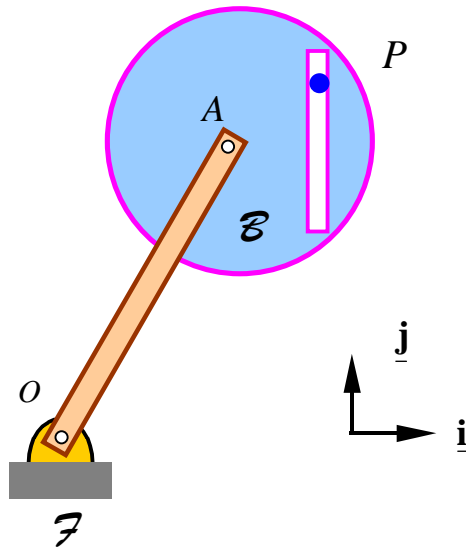
Calculate the velocity and acceleration of the particle P that is moving with constant speed v_0 relative to the disc along the groove. The bar OA is rotating with a constant counter-clockwise rotation of ω_1 relative to the ground and the disc D is rotating at a constant CCW rate of ω_2 relative to OA . The length of OA is l .



SOLUTION:

To calculate the velocity we have

$$[\underline{v}_{P/O}]_{\mathcal{F}} = [\underline{v}_{P/A}]_{\mathcal{B}} + \underline{\omega}_{\mathcal{B}/\mathcal{F}} \times \underline{r}_{P/A} + [\underline{v}_{A/O}]_{\mathcal{F}} \quad (1)$$



Note that

$$\begin{aligned} [\underline{v}_{P/A}]_{\mathcal{B}} &= v_o \underline{j} \\ \underline{\omega}_{\mathcal{B}/\mathcal{F}} &= (\omega_1 + \omega_2) \underline{k} \\ \underline{r}_{P/A} &= b \underline{i} + h \underline{j} \end{aligned}$$

$$\begin{aligned} \underline{\omega}_{\mathcal{B}/\mathcal{F}} \times \underline{r}_{P/A} &= (\omega_1 + \omega_2) \underline{k} \times (b \underline{i} + h \underline{j}) \\ &= (\omega_1 + \omega_2)(b \underline{j} - h \underline{i}) \end{aligned}$$

$$\begin{aligned} [\underline{v}_{A/O}]_{\mathcal{F}} &= \underline{\omega}_{OA} \times \underline{r}_{A/O} = (\omega_1 \underline{k}) \times (l \cos \theta \underline{i} + l \sin \theta \underline{j}) \\ &= \omega_1 (-l \sin \theta \underline{i} + l \cos \theta \underline{j}) \end{aligned}$$

Substitution into (1) to get the velocity of P, this gives

$$\begin{aligned} [\underline{v}_{P/O}]_{\mathcal{F}} &= v_o \underline{j} + (\omega_1 + \omega_2)(b \underline{j} - h \underline{i}) + \omega_1 (l \cos \theta \underline{j} - l \sin \theta \underline{i}) \\ &= -[(\omega_1 + \omega_2)h + l \sin \theta] \underline{i} + [v_o + (\omega_1 + \omega_2)b + \omega_1 l \cos \theta] \underline{j} \end{aligned}$$

To get the acceleration we have

$$\begin{aligned} [\underline{a}_{P/O}]_{\mathcal{F}} &= [\underline{a}_{P/A}]_{\mathcal{E}} + 2\underline{\omega}_{\mathcal{E}/\mathcal{F}} \times [\underline{v}_{P/A}]_{\mathcal{E}} + \underline{\alpha}_{\mathcal{E}/\mathcal{F}} \times \underline{r}_{P/A} \\ &\quad + \underline{\omega}_{\mathcal{E}/\mathcal{F}} \times (\underline{\omega}_{\mathcal{E}/\mathcal{F}} \times \underline{r}_{P/A}) + [\underline{a}_{A/O}]_{\mathcal{F}} \quad (2) \end{aligned}$$

Note that

$$\begin{aligned} [\underline{a}_{P/A}]_{\mathcal{E}} &= 0 \\ 2\underline{\omega}_{\mathcal{E}/\mathcal{F}} \times [\underline{v}_{P/A}]_{\mathcal{E}} &= 2(\omega_1 + \omega_2) \underline{k} \times (v_o \underline{j}) \\ &= 2(\omega_1 + \omega_2) v_o \underline{i} \\ \underline{\alpha}_{\mathcal{E}/\mathcal{F}} &= 0 \Rightarrow \underline{\alpha}_{\mathcal{E}/\mathcal{F}} \times \underline{r}_{P/A} = 0 \\ \underline{\omega}_{\mathcal{E}/\mathcal{F}} \times (\underline{\omega}_{\mathcal{E}/\mathcal{F}} \times \underline{r}_{P/A}) &= (\omega_1 + \omega_2) \underline{k} \times [(\omega_1 + \omega_2)(b \underline{j} - h \underline{i})] \\ &= -(\omega_1 + \omega_2)^2 (h \underline{j} + b \underline{i}) \end{aligned}$$

$$\begin{aligned} [\underline{a}_{A/O}]_{\mathcal{F}} &= \underline{\alpha}_{OA} \times \underline{r}_{A/O} + \underline{\omega}_{OA} \times (\underline{\omega}_{OA} \times \underline{r}_{A/O}) \\ &= 0 + \omega_1 \underline{k} \times \omega_1 (l \cos \theta \underline{j} - l \sin \theta \underline{i}) \\ &= \omega_1^2 (l \sin \theta \underline{j} + l \cos \theta \underline{i}) \end{aligned}$$

Substitution into (2) gives the acceleration of P as

$$\begin{aligned} [\underline{a}_{P/O}]_{\mathcal{F}} &= 2(\omega_1 + \omega_2) v_o \underline{i} - (\omega_1 + \omega_2)^2 (b \underline{i} + h \underline{j}) \\ &\quad \omega_1^2 (l \sin \theta \underline{j} + l \cos \theta \underline{i}) \\ &= -[2(\omega_1 + \omega_2) v_o + (\omega_1 + \omega_2)^2 b + \omega_1^2 l \cos \theta] \underline{i} \\ &\quad - [(\omega_1 + \omega_2)^2 h + \omega_1^2 l \sin \theta] \underline{j} \end{aligned}$$