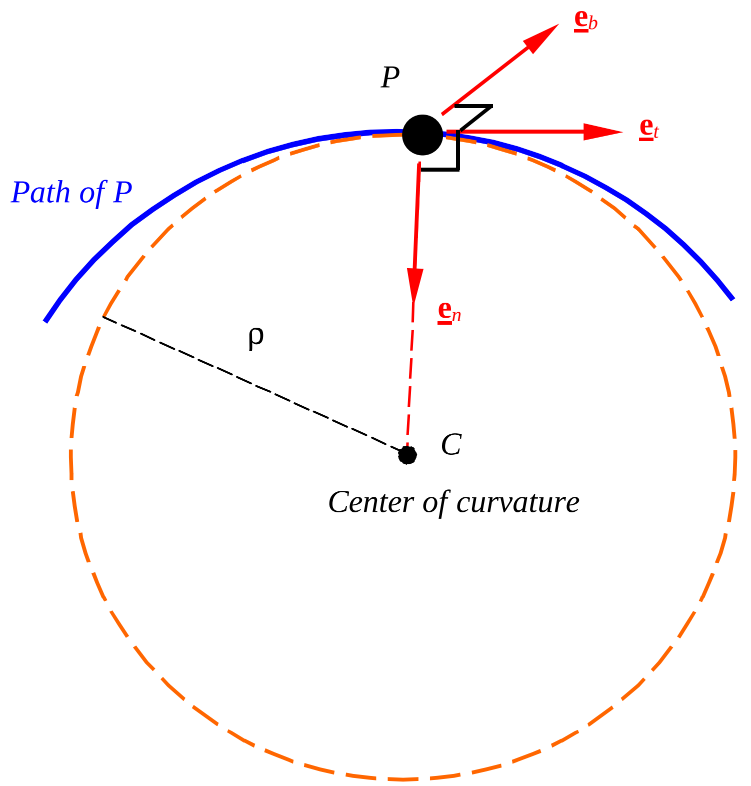
MÜHENDİSLİK MEKANİĞİ

9. HAFTA Düzlemsel Hareket
Normal ve Teğetsel Koordinatlar

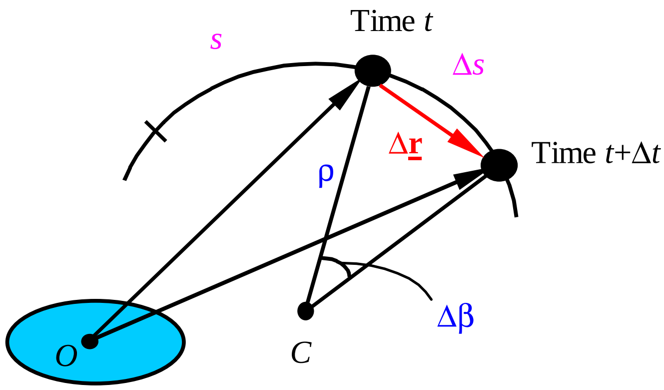


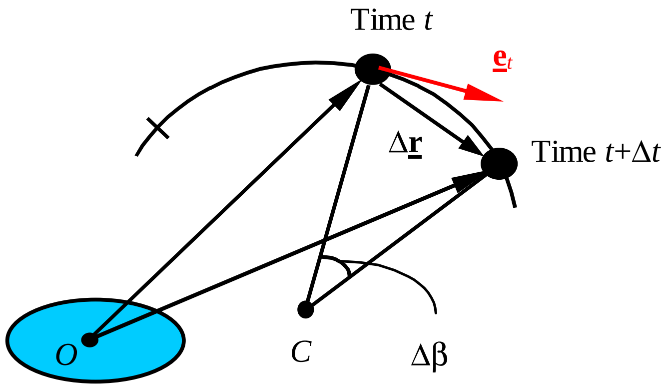
$\underline{\mathbf{e}}_t$: yörüngenin birim teğet vektörü

$\underline{\mathbf{e}}_n$: yörüngenin birim normal vektörü

$\underline{\mathbf{e}}_b$: Yörüngenin normaline ve teğetine dik vektörü ($\underline{e}_b = \underline{e}_t \times \underline{e}_n$)

Hız:





$$\left.\begin{array}{l} \underline{v} = \dfrac{d\underline{r}}{dt} = \lim\limits_{\Delta t\to 0}\dfrac{\Delta \underline{r}}{\Delta t} \\ |\Delta \underline{r}| \to \Delta s \\ \Delta \underline{r} \to \Delta s \underline{e}_t \end{array}\right\} \to \underline{v} = \lim\limits_{\Delta t\to 0}\frac{\Delta s}{\Delta t}\underline{e}_t = \dot{s}\underline{e}_t = v\underline{e}_t$$

$$\Delta s \to \rho\Delta\beta \quad \Rightarrow \frac{\Delta s}{\Delta t} \to \rho\frac{\Delta\beta}{\Delta t} \quad \Rightarrow v = \dot{s} = \rho\dot{\beta} \quad \Rightarrow \underline{\mathbf{v}} = \rho\dot{\beta}\underline{\mathbf{e}}_t$$

$$\underline{\mathbf{v}} = v_t\underline{\mathbf{e}}_t + v_n\underline{\mathbf{e}}_n \quad \Rightarrow \begin{cases} v_t = v = \dot{s} = \rho\dot{\beta} \\ v_n = 0 \end{cases}$$

Örnek 1: Normal & Teğetsel koordinatlar

$\underline{\mathbf{v}} = 3t\underline{\mathbf{i}} + 4t\,\underline{\mathbf{j}}$, hız veriliyor

Teğetsel hızını bulun $\underline{\mathbf{e}}_t$.

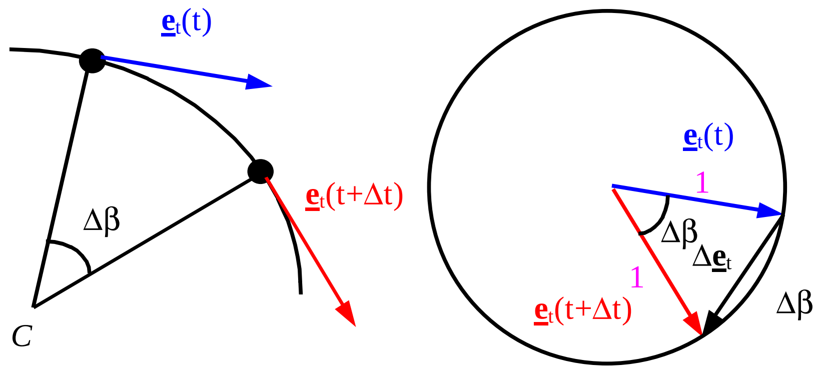
Çözüm:

$$v \equiv \sqrt{v_x^2 + v_y^2}$$

$$v = \sqrt{(3t)^2 + (4t)^2} = 5t$$

$$\underline{\mathbf{e}}_t \equiv \frac{\underline{\mathbf{v}}}{v} = \frac{3t\underline{\mathbf{i}} + 4t\underline{\mathbf{j}}}{5t} = \frac{3}{5}\underline{\mathbf{i}} + \frac{4}{5}\underline{\mathbf{j}}$$

İvme:



$$\left.\begin{array}{l} \dot{\underline{\mathbf{e}}}_t = \lim\limits_{\Delta t \to 0} \dfrac{\Delta \underline{\mathbf{e}}_t}{\Delta t} \\ \Delta \underline{\mathbf{e}}_t \to \Delta\beta\, \underline{\mathbf{e}}_n \end{array}\right\} \Rightarrow \dot{\underline{\mathbf{e}}}_t = \dot{\beta}\,\underline{\mathbf{e}}_n = \frac{v}{\rho}\underline{\mathbf{e}}_n$$

$$\left.\begin{array}{l} \underline{\mathbf{a}} = \dfrac{d\underline{\mathbf{v}}}{dt} = \dot{v}\underline{\mathbf{e}}_t + v\dot{\underline{\mathbf{e}}}_t \\ \dot{\underline{\mathbf{e}}}_t = \dfrac{v}{\rho}\underline{\mathbf{e}}_n \end{array}\right\} \Rightarrow \underline{\mathbf{a}} = \dot{v}\underline{\mathbf{e}}_t + \frac{v^2}{\rho}\underline{\mathbf{e}}_n$$

$$\underline{\mathbf{a}} = a_t\underline{\mathbf{e}}_t + a_n\underline{\mathbf{e}}_n \Rightarrow \begin{cases} a_t = \dot{v} = \ddot{s} & \text{tangential acceleration} \\ a_n = \dfrac{v^2}{\rho} = \rho\dot{\beta}^2 & \text{normal acceleration} \end{cases}$$

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{\dot{v}^2 + \frac{v^4}{\rho^2}}$$

Örnek 2: Normal & Teğetsel Koordinatlar

$v = 5t,\ a_n = 6t,$

Hız ve ivme verilmiş

Teğetsel ivme a_t ve eğrilik yarıçapını bulunuz ρ .

Çözüm:

$$a_t \equiv \dot{v} = \frac{d(5t)}{dt} = 5$$

$$a_n \equiv \frac{v^2}{\rho}$$

$$\rho = \frac{v^2}{a_n} = \frac{25t^2}{6t} = \frac{25t}{6}$$

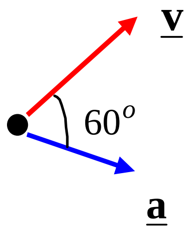
Örnek 3: Normal & Teğetsel Koordinatlar

$v = 5\, m/s,\ a = 10\, m/s^2$,

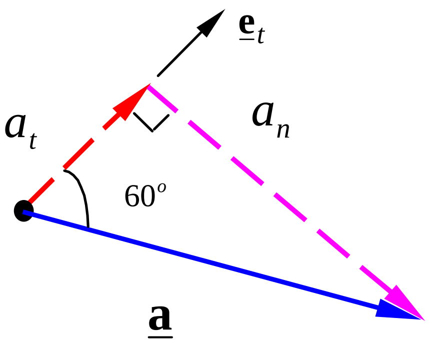
verilmiş

$a_{t,}\ a_n$ and ρ .

Değerlerini hesaplayın



Çözüm:

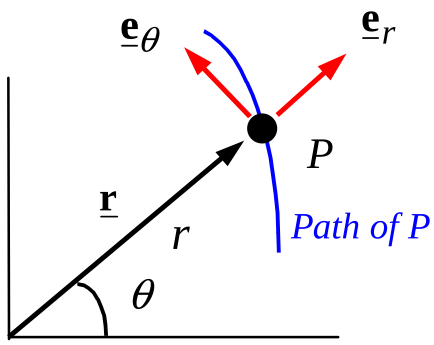


$$a_t = a\cos 60 = 10 \times 0.5 = 5\, m/s^2$$

$$a_n = a\sin 60 = 10 \times \frac{\sqrt{3}}{2} = 5\sqrt{3}$$

$$a_n = \frac{v^2}{\rho} \Rightarrow \rho = \frac{v^2}{a_n} = \frac{5^2}{5\sqrt{3}} = \frac{5\sqrt{3}}{3} m$$

Polar Koordinatlar



Pozisyon:

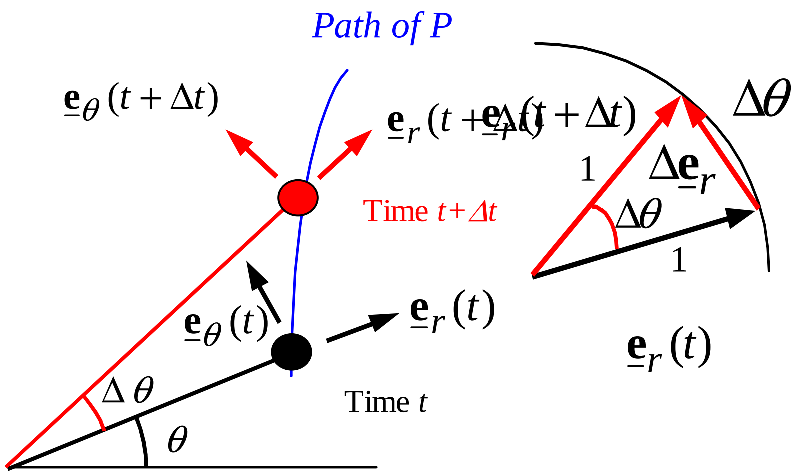
$$\boxed{\underline{\mathbf{r}} = r\underline{\mathbf{e}}_r}$$

Hız:

$$\underline{\mathbf{v}} = \frac{d\underline{\mathbf{r}}}{dt} = \dot{r}\underline{\mathbf{e}}_r + r\dot{\underline{\mathbf{e}}}_r$$

$$\dot{\underline{\mathbf{e}}}_r = \lim_{\Delta t \to 0} \frac{\Delta \underline{\mathbf{e}}_r}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} \underline{\mathbf{e}}_\theta = \dot{\theta}\, \underline{\mathbf{e}}_\theta$$

$$\boxed{\Rightarrow \underline{\mathbf{v}} = \dot{r}\underline{\mathbf{e}}_r + r\dot{\theta}\underline{\mathbf{e}}_\theta}$$



$$\Rightarrow \begin{cases} v_r = \dot{r} \\ \\ v_\theta = r\dot{\theta} \end{cases}$$

Örnek 1: POLAR KOORDİNATLAR

Bir parçacığın polar koordinatlardaki aşağıdaki gibi verilmiş

$r = r_o + b\sin(\theta\ ) \ ,\ \theta\ = 2\pi\ \mathrm{t}^2$,

Bileşke hızı hesaplayın

Çözüm:

$$\dot{\theta} = \frac{d\theta}{dt} = 4\pi\ t$$

$$\dot{r} = b\frac{d(\sin\theta\ )}{dt} = b\frac{d(\sin\theta\ )}{d\theta}\frac{d\theta}{dt} = b\cos\theta\ \dot{\theta}$$

$$= b\cos(2\pi\ t^2)4\pi\ t$$

$$v_r = \dot{r} = 4\pi bt\cos(2\pi t^2)$$

$$v_\theta = r\dot{\theta} = [r_o + b\sin(2\pi t^2)]4\pi t$$

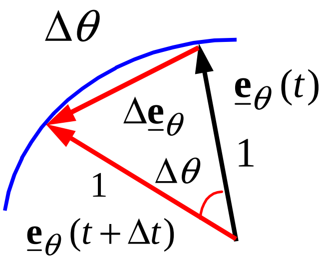
$$v = \sqrt{v_r^2 + v_\theta^2}$$

İvme:

$$\underline{\mathbf{a}} = \frac{d\underline{\mathbf{v}}}{dt} = \ddot{r}\underline{e}_r + \dot{r}\underline{\dot{\mathbf{e}}}_r + \dot{r}\dot{\theta}\underline{\mathbf{e}}_\theta + r\ddot{\theta}\underline{\mathbf{e}}_\theta + r\dot{\theta}\underline{\dot{\mathbf{e}}}_\theta$$

$$\underline{\dot{\mathbf{e}}}_r = \dot{\theta}\,\underline{\mathbf{e}}_\theta$$

$$\underline{\dot{\mathbf{e}}}_\theta = \lim_{\Delta t \to 0} \frac{\Delta \underline{\mathbf{e}}_\theta}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t}(-\underline{\mathbf{e}}_r) = -\dot{\theta}\underline{\mathbf{e}}_r$$



$$\boxed{\Rightarrow \underline{\mathbf{a}} = (\ddot{r} - r\dot{\theta}^2)\underline{\mathbf{e}}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\underline{\mathbf{e}}_\theta}$$

$$\Rightarrow \begin{cases} a_r = \ddot{r} - r\dot{\theta}^2 \\ a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} \end{cases}$$

Örnek 2: POLAR KOORDİNATLAR

$r = a\cos\theta$ and $\theta = bt$,

Şeklinde veriliyor

a_r, a_θ, and a.

Değerlerini bulun

Çözüm:

$$\dot{\theta} = \frac{d\theta}{dt} = b$$

$$\dot{r} = \frac{dr}{dt} = \frac{d(a\cos\theta)}{d\theta}\frac{d\theta}{dt} = -ab\sin\theta$$

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{db}{dt} = 0$$

$$\ddot{r} = \frac{d\dot{r}}{dt} = \frac{d(-ab\sin\theta)}{d\theta}\frac{d\theta}{dt} = -ab^2\cos\theta$$

$$\begin{aligned} a_r &= \ddot{r} - r\dot{\theta}^{\,2} \\ &= -ab^2\cos\theta - a\cos\theta\, b^2 \\ &= -2ab^2\cos\theta \end{aligned}$$

$$\begin{aligned} a_\theta &= 2\dot{r}\dot{\theta} + r\ddot{\theta} \\ &= 2(-ab\sin\theta)b + 0 \\ &= -2ab^2\sin\theta \end{aligned}$$

$$a = \sqrt{a_r^2 + a_\theta^2} = 2\sqrt{2}ab^2$$

Lemma:

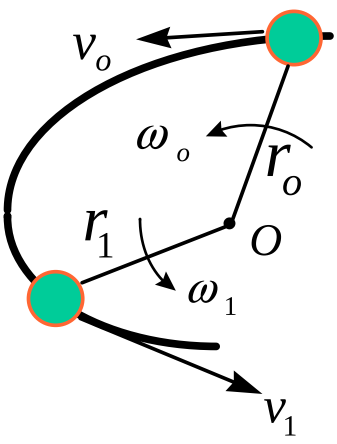
$$\boxed{a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})}$$

Örnek 3: POLAR KOORDİNATLAR

Bir parçacığın hareketi $a_\theta = 0$,
Olarak veriliyor.

$r = r_1$ anındaki açısal hızı ω_1 bulun

Verilen başlangıç koşulları $\dot{\theta} = \omega_o$ at $r = r_o$.
Şeklinde veriliyor



Çözüm:

$$a_\theta = 0 = \frac{1}{r}\frac{dr}{dt}(r^2\dot{\theta})$$

$$\Rightarrow \frac{d}{dr}(r^2\dot{\theta}) = 0$$

$$\Rightarrow r^2\dot{\theta} = \text{constant}$$

$$\dot{\theta} = \omega_o \text{ at } r = r_o \Rightarrow r^2\dot{\theta} = r_o^2\omega_o$$

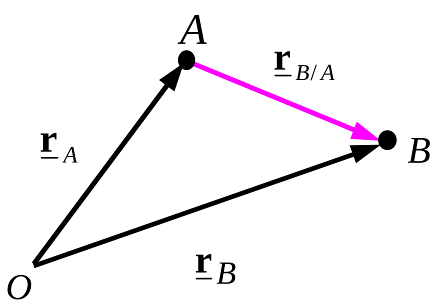
$$\Rightarrow \omega_1 = \frac{r_o^2\omega_o}{r_1^2}$$

İki Parçacığın rölatif hareketi

$$\underline{\mathbf{r}}_{B/A} = \underline{\mathbf{r}}_B - \underline{\mathbf{r}}_A$$

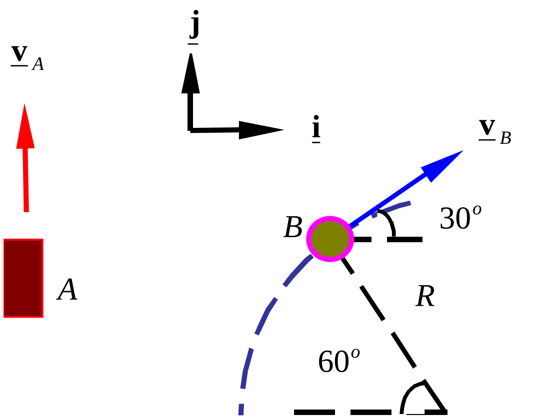
$$\underline{\mathbf{v}}_{B/A} = \underline{\mathbf{v}}_B - \underline{\mathbf{v}}_A$$

$$\underline{\mathbf{a}}_{B/A} = \underline{\mathbf{a}}_B - \underline{\mathbf{a}}_A$$

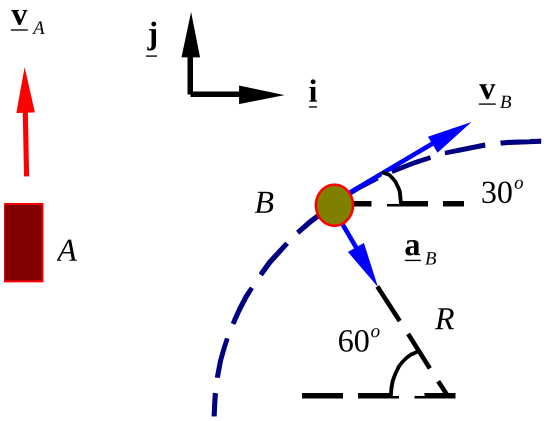


Örnek 1: Rölatif Hareket

A parçacığı bir doğru üzerinde $v_A = a_o t$, hızıyla hareket ediyor ve B parçacığı R yarıçaplı daire üzerinde $v_B = v_o$, sabit hızıyla hareket ediyor $\underline{\mathbf{v}}_{B/A}$ ve $\underline{\mathbf{a}}_{B/A}$. Değerlerini bulun



Çözüm:



$$\underline{\mathbf{v}}_A = a_o t\,\underline{\mathbf{j}}$$

$$\underline{\mathbf{v}}_B = v_o \cos 30\underline{\mathbf{i}} + v_o \sin 30\underline{\mathbf{j}} = \frac{\sqrt{3}}{2} v_o \underline{\mathbf{i}} + \frac{1}{2} v_o \underline{\mathbf{j}}$$

$$\underline{\mathbf{v}}_{B/A} = \underline{\mathbf{v}}_B - \underline{\mathbf{v}}_A = \frac{\sqrt{3}}{2} v_0 \underline{\mathbf{i}} + (\frac{1}{2} v_0 - a_o t)\underline{\mathbf{j}}$$

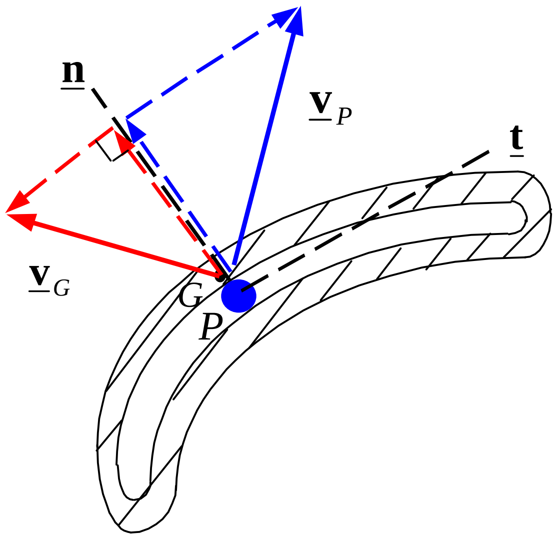
$$a_{Bt} \equiv \dot{v}_B = 0, \quad a_{Bn} \equiv \frac{v_B^2}{R} \Rightarrow a_B = \frac{v_B^2}{R} = \frac{v_0^2}{R}$$

$$\underline{\mathbf{a}}_B = a_B \sin 30\underline{\mathbf{i}} - a_B \cos 30\underline{\mathbf{j}} = \frac{1}{2} a_B \underline{\mathbf{i}} - \frac{\sqrt{3}}{2} a_B \underline{\mathbf{j}}$$

$$\underline{\mathbf{a}}_A = \dot{\underline{\mathbf{v}}}_A = a_o \underline{\mathbf{j}}$$

$$\underline{\mathbf{a}}_{B/A} = \underline{\mathbf{a}}_B - \underline{\mathbf{a}}_A = \frac{1}{2}\frac{v_o^2}{R}\underline{\mathbf{i}} - \left(\frac{\sqrt{3}}{2}\frac{v_o^2}{R} - a_o\right)\underline{\mathbf{j}}$$

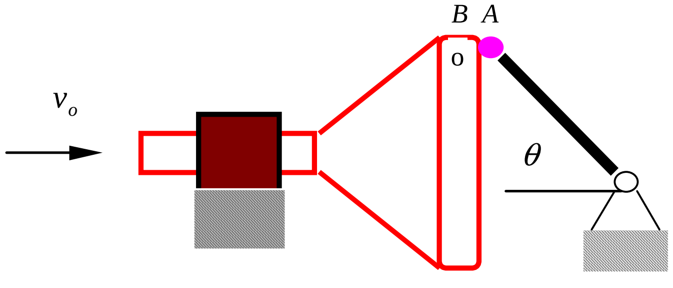
Hareketli eksen takımında rölatif haraket



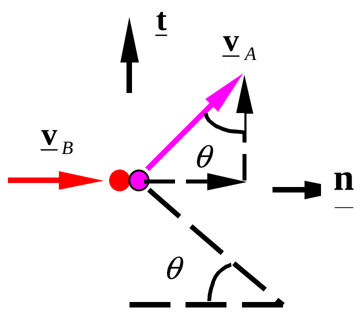
$$\underline{\mathbf{v}}_P \circ \underline{\mathbf{n}} = \underline{\mathbf{v}}_G \circ \underline{\mathbf{n}}$$

Örnek 2: Rölatif Hareket

B ve A parçacıkları arasındaki hız ilişkisini bulunuz

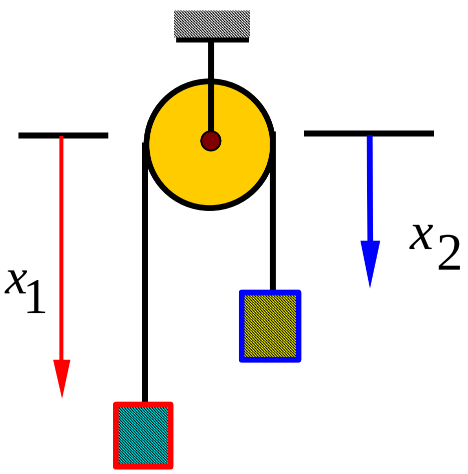


Çözüm:



$$\left.\begin{array}{l} \underline{\mathbf{v}}_A \circ \underline{\mathbf{n}} = v_A \sin\theta \\ \underline{\mathbf{v}}_B \circ \underline{\mathbf{n}} = v_o \end{array}\right\} \Rightarrow v_A \sin\theta = v_o$$

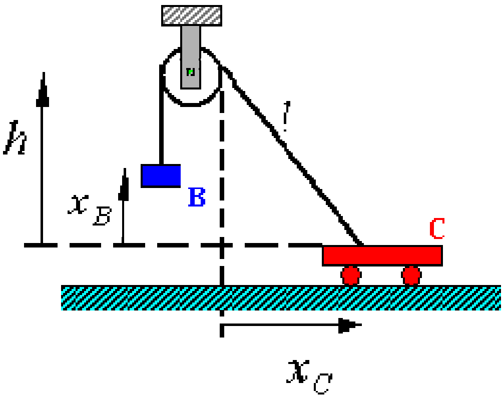
Makaralar:



$$x_1 + x_2 = \text{costant} \Rightarrow \dot{x}_1 + \dot{x}_2 = 0$$
$$\Rightarrow \dot{x}_1 = -\dot{x}_2 \Rightarrow \ddot{x}_1 = -\ddot{x}_2$$

Örnek 3: Rölatif Hareket

Araba ve bloğun hızları arasındaki bağıntıyı bulun



Çözüm:

Kablonun uzunluğu sabit $= l + h - x_B$

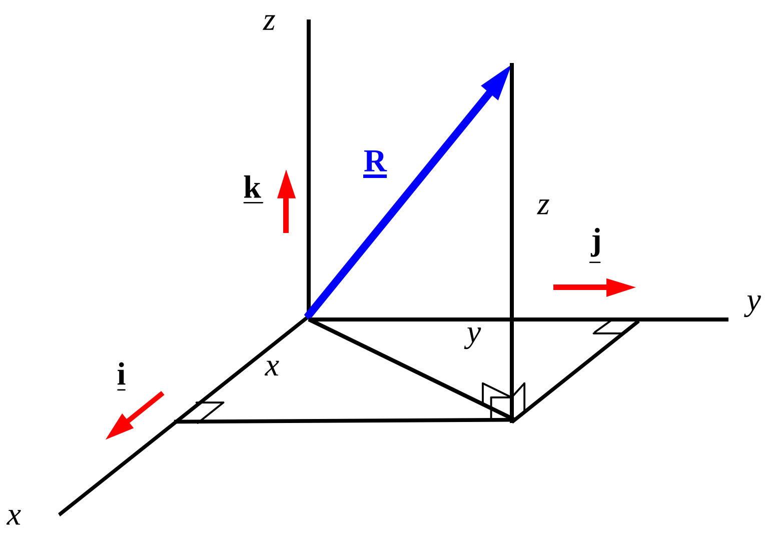
$$\Rightarrow \sqrt{x_C^2 + h^2} + h - x_B = \text{constant}$$

Her iki tarafın türevi alınarak hesaplanabilir.

$$\frac{x_C \dot{x}_C}{\sqrt{x_C^2 + h^2}} - \dot{x}_B = 0$$

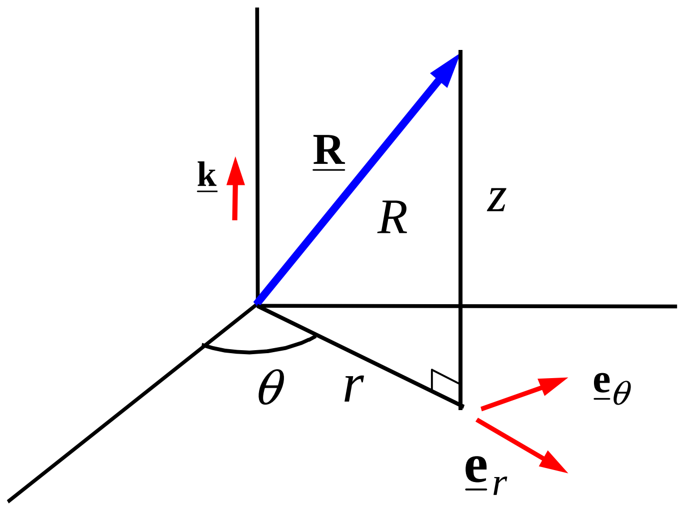
$$\Rightarrow \dot{x}_B = \frac{x_C \dot{x}_C}{\sqrt{x_C^2 + h^2}}$$

Uzayda eğrisel hareket



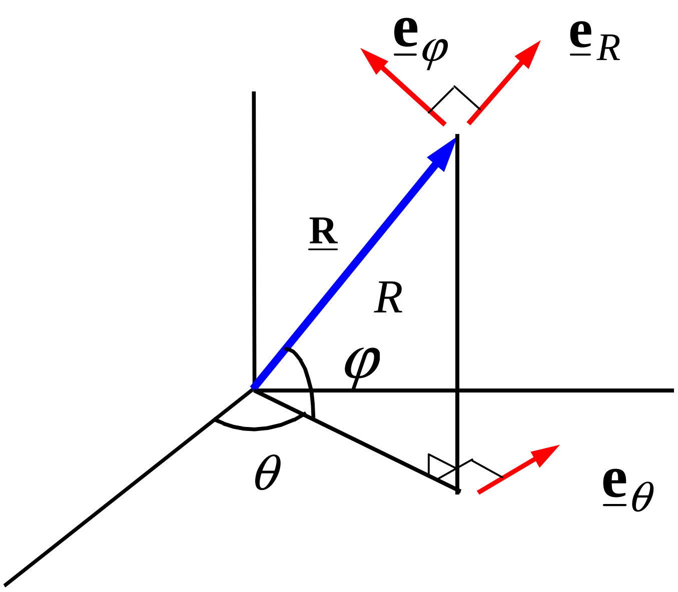
Kartezyen Koordinatlar (x,y,z)**:**

$$\underline{\mathbf{R}} = x\underline{\mathbf{i}} + y\underline{\mathbf{j}} + z\underline{\mathbf{k}}$$
$$\underline{\mathbf{v}} = \underline{\dot{\mathbf{R}}} = \dot{x}\underline{\mathbf{i}} + \dot{y}\underline{\mathbf{j}} + \dot{z}\underline{\mathbf{k}}$$
$$\underline{\mathbf{a}} = \underline{\ddot{\mathbf{R}}} = \ddot{x}\underline{\mathbf{i}} + \ddot{y}\underline{\mathbf{j}} + \ddot{z}\underline{\mathbf{k}}$$



Silindirik Koordinatlar (r, θ, z):

$$\underline{\mathbf{R}} = r\underline{\mathbf{e}}_r + z\underline{\mathbf{k}}$$
$$\underline{\mathbf{v}} = \underline{\dot{\mathbf{R}}} = \dot{r}\underline{\mathbf{e}}_r + r\dot{\theta}\underline{\mathbf{e}}_\theta + \dot{z}\underline{\mathbf{k}}$$
$$\underline{\mathbf{a}} = \underline{\ddot{\mathbf{R}}} = (\ddot{r} - r\dot{\theta}^2)\underline{\mathbf{e}}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\underline{\mathbf{e}}_\theta + \ddot{z}\underline{\mathbf{k}}$$



Küresel Koordinatlar (R,θ,φ):

$$\underline{\mathbf{R}} = R\underline{\mathbf{e}}_R$$

$$\underline{\mathbf{v}} = R\underline{\mathbf{e}}_R + R\dot{\theta}\cos(\varphi)\underline{\mathbf{e}}_\theta + R\dot{\varphi}\underline{\mathbf{e}}_\varphi$$

$$a = a_R\underline{\mathbf{e}}_R + a_\theta\underline{\mathbf{e}}_\theta + a_\varphi\underline{\mathbf{e}}_\varphi$$

$$a_R = \ddot{R} - R\dot{\varphi}^2 - R\dot{\theta}^2\cos^2(\varphi)$$

$$a_\theta = \frac{\cos(\varphi)}{R}\frac{d}{dt}(R^2\dot{\theta}) - 2R\dot{\theta}\dot{\varphi}\sin(\varphi)$$

$$a_\varphi = \frac{1}{R}\frac{d}{dt}(R^2\dot{\phi}) + R\dot{\theta}^2\sin(\varphi)\cos(\varphi)$$