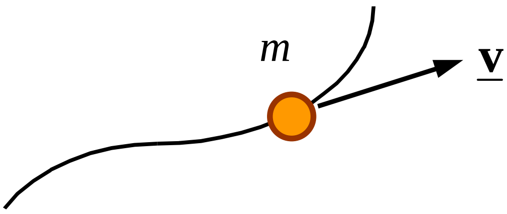
MÜHENDİSLİK MEKANİĞİ

12. HAFTA
İMPULS- MOMENTUM-ÇARPIŞMA

Linear momentum of a particle: The symbol $\underline{\mathbf{L}}$ denotes the linear momentum and is defined as the mass times the velocity of a particle.

$$\boxed{\underline{\mathbf{L}} \equiv m\underline{\mathbf{v}}}$$



ÖRNEK 1: THE LINEAR IMPULSE-MOMENTUM RELATION

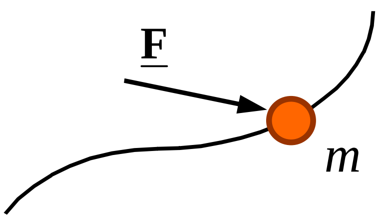
Calculate the linear momentum of a particle of mass m=10 kg which has a velocity of $\underline{\mathbf{v}} = 3\underline{\mathbf{i}} + 4\underline{\mathbf{j}} + 3\underline{\mathbf{k}}$ m/s.

Çözüm:

$$\underline{\mathbf{L}} \equiv m\underline{\mathbf{v}} = 10(3\underline{\mathbf{i}} + 4\underline{\mathbf{j}} + 3\underline{\mathbf{k}})$$

$$= 30\underline{\mathbf{i}} + 40\underline{\mathbf{j}} + 30\underline{\mathbf{k}} \quad \frac{kg-m}{s}$$

Impulse of a force from time t_1 to t_2: The integral of the force over the time interval of concern is its impulse. The impulse of a force is a vector given by the integral



$$\underline{\mathbf{I}} \equiv \int_{t_1}^{t_2} \underline{\mathbf{F}} dt$$

ÖRNEK 2: THE LINEAR IMPULSE-MOMENTUM RELATION

Calculate the impulse of the force $\underline{\mathbf{F}} = 2t\,\underline{\mathbf{i}} + 3t^2\,\underline{\mathbf{j}}$ N from time t = 0 s to time t = 2 s.

Çözüm:

$$\underline{\mathbf{I}} = \int_{t_1}^{t_2} \underline{\mathbf{F}} dt = \int_0^2 (2t\,\underline{\mathbf{i}} + 3t^2\,\underline{\mathbf{j}}) dt = (t^2\,\underline{\mathbf{i}} + t^3\,\underline{\mathbf{j}})\Big|_{t=0}^{t=2}$$
$$= 4\,\underline{\mathbf{i}} + 8\,\underline{\mathbf{j}} \quad N - s$$

Newton's 2nd law: Newton's second law states that the resultant of all forces applied on a particle is equal to the rate of change of linear momentum of the particle.

$$\boxed{\sum \underline{\mathbf{F}} = \dot{\underline{\mathbf{L}}}}$$

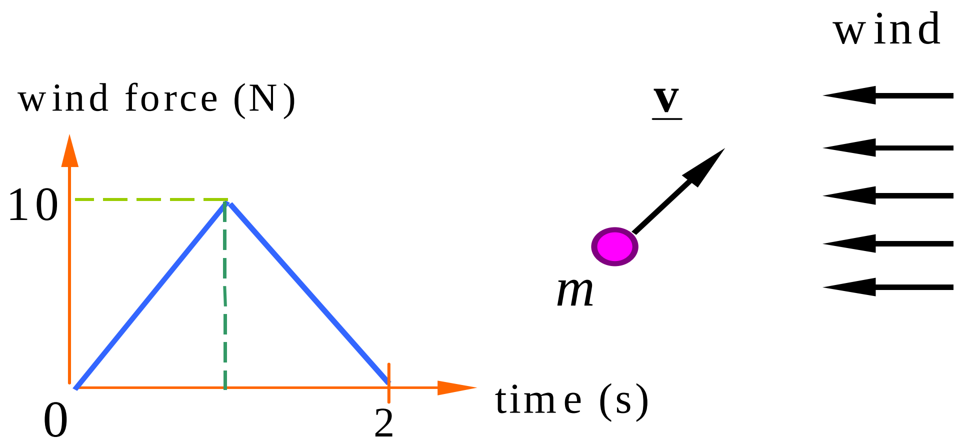
This reduces to the more familiar statement of this law if one notes that in Newtonian mechanics it is assumed that mass is constant, and, therefore, $\dot{\underline{\mathbf{L}}} = \dot{m}\underline{\mathbf{v}} + m\dot{\underline{\mathbf{v}}} = m\underline{\mathbf{a}}$.

The linear impulse-momentum relation: Integration of Newton's 2nd law over the time interval from t_1 to t_2 results in

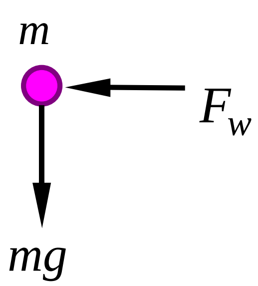
$$\int_{t_1}^{t_2} \sum \underline{\mathbf{F}}\, dt = \int_{t_1}^{t_2} \dot{\underline{\mathbf{L}}} dt \Rightarrow \int_{t_1}^{t_2} \sum \underline{\mathbf{F}}\, dt = \underline{\mathbf{L}}(t_2) - \underline{\mathbf{L}}(t_1) \Rightarrow$$

$$\boxed{\underline{\mathbf{L}}_1 + \int_{t_1}^{t_2} \sum \underline{\mathbf{F}}\, dt = \underline{\mathbf{L}}_2}$$

ÖRNEK 3: THE LINEAR IMPULSE-MOMENTUM RELATION



A projectile of mass *m*=10 *kg* having an initial velocity of $\underline{\mathbf{v}} = 3\underline{\mathbf{i}} + 4\underline{\mathbf{j}}$ m/s is subjected to a horizontal wind force as shown in the figure. Calculate the projectile's velocity just after the wind has ended.



Çözüm:

Linear impulse-momentum relation:

$$\underline{\mathbf{L}}_1 + \int_{t_1}^{t_2} \sum \underline{\mathbf{F}} dt = \underline{\mathbf{L}}_2 \quad (1)$$

$$\underline{\mathbf{L}}_1 = m\underline{\mathbf{v}}_1 = 10(3\underline{\mathbf{i}} + 4\underline{\mathbf{j}}) = 30\underline{\mathbf{i}} + 40\underline{\mathbf{j}} \quad \frac{kg-m}{s}$$

$$\int_{t_1}^{t_2} \sum \underline{\mathbf{F}} dt = \int_{t_1}^{t_2} (-F_w \underline{\mathbf{i}} - mg\,\underline{\mathbf{j}}) dt = -\int_0^2 F_w dt\,\underline{\mathbf{i}} - mg\int_0^2 dt\,\underline{\mathbf{j}}$$

$$= -10\underline{\mathbf{i}} - 10(9.81)(2)\underline{\mathbf{j}} = -10\underline{\mathbf{i}} - 196.2\underline{\mathbf{j}} N-s$$

$$\underline{\mathbf{L}}_2 = m\underline{\mathbf{v}}_2 = 10\underline{\mathbf{v}}_2$$

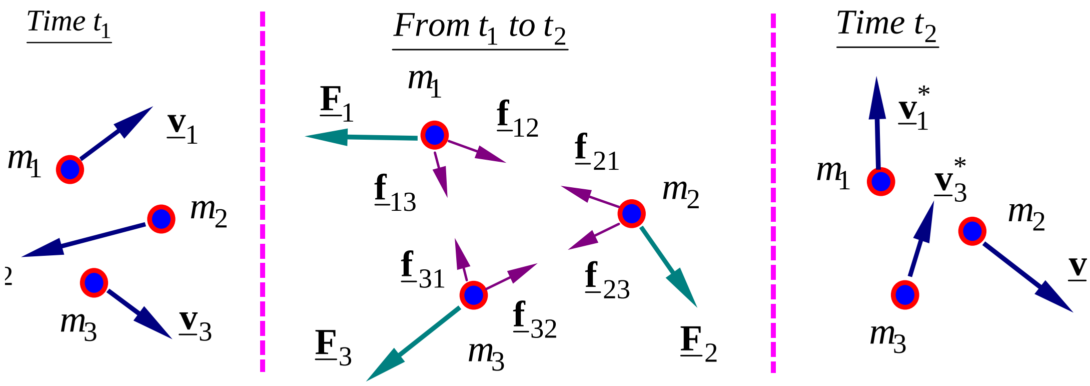
Substitution into (1) gives:

$$30\underline{\mathbf{i}} + 40\underline{\mathbf{j}} - 10\underline{\mathbf{i}} - 196.2\underline{\mathbf{j}} = 10\underline{\mathbf{v}}_2$$

$$\Rightarrow \underline{\mathbf{v}}_2 = 2\underline{\mathbf{i}} - 15.6\underline{\mathbf{j}} \quad \frac{m}{s}$$

Therefor, the linear momentum of a particle is changed by the impulse of the resultant force on the particle. There will be conservation of linear momentum only if the impulse of the resultant force is zero.

For a system of particles: Consider the system of particles shown below. Each particle in the system has a mass m_i and at time t_1 has a velocity $\underline{\mathbf{v}}_i$, and from time t_1 to time t_2 is acted upon by the resultant external force $\underline{\mathbf{F}}_i$ and the internal forces of interaction between the particles of $\underline{\mathbf{f}}_{ij}$. As a result of the impulse of the internal and external forces on each particle, at time t_2 the particle with mass m_i has a velocity $\underline{\mathbf{v}}^*_i$.



Writing the impulse momentum equation for each particle and adding them up, considering that Newton's 3rd law requires that $\underline{\mathbf{f}}_{ij} = \underline{\mathbf{f}}_{ji}$, we get for n particles

$$for\ \ m_1 : m_1\underline{\mathbf{v}}_1 + \int_{t_1}^{t_2} (\underline{\mathbf{F}}_1 + \underline{\mathbf{f}}_{12} + \underline{\mathbf{f}}_{13} + \ldots)dt = m_1\underline{\mathbf{v}}_1^*$$

$$for\ \ m_2 : m_2\underline{\mathbf{v}}_2 + \int_{t_1}^{t_2} (\underline{\mathbf{F}}_2 + \underline{\mathbf{f}}_{21} + \underline{\mathbf{f}}_{22} + \ldots)dt = m_2\underline{\mathbf{v}}_2^*$$

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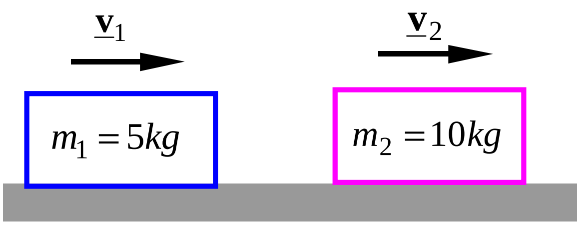
$$for\ \ m_n : m_n\underline{\mathbf{v}}_n + \int_{t_1}^{t_2} (\underline{\mathbf{F}}_n + \underline{\mathbf{f}}_{n1} + \underline{\mathbf{f}}_{n2} + \ldots)dt = m_n\underline{\mathbf{v}}_n^*$$

$$\sum_{i=1}^{n} m_i\underline{\mathbf{v}}_i + \int_{t_1}^{t_2} (\sum_{i=1}^{n} \underline{\mathbf{F}}_i)dt = \sum_{i=1}^{n} m_i\underline{\mathbf{v}}_i^*$$

Therefore, the total linear momentum of the system is changed by the impulse of the external forces. Recalling the relation $m\underline{\mathbf{v}}_{CM} = \sum m\underline{\mathbf{v}}$, we can rewrite this equation as

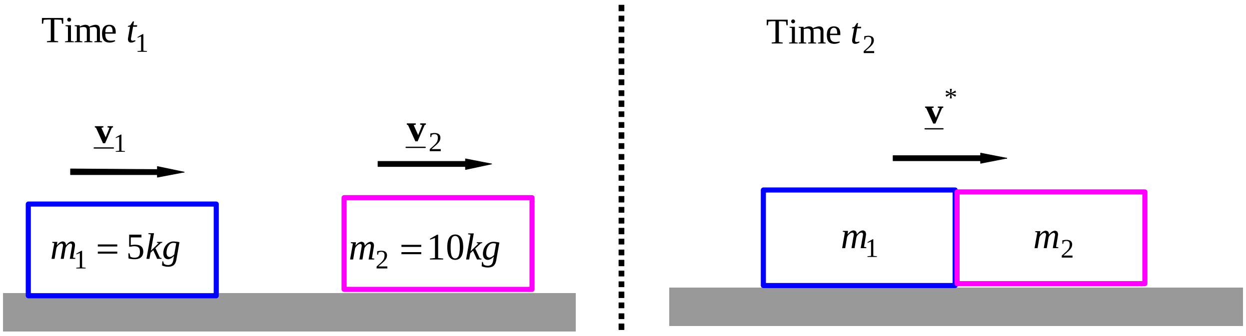
$$m\underline{\mathbf{v}}_{CM} + \int_{t_1}^{t_2} \sum \underline{\mathbf{F}}_{ext} dt = m\underline{\mathbf{v}}_{CM}^{*}$$

ÖRNEK 4: THE LINEAR IMPULSE-MOMENTUM RELATION



The two particles are moving at velocity $\underline{\mathbf{v}}_1 = 10\underline{\mathbf{i}}\ m/s$ and $\underline{\mathbf{v}}_2 = 5\underline{\mathbf{i}}\ m/s$ just before they collide and become connected. Calculate the velocity of the system after the collision.

Solution:



$$\sum m\underline{\mathbf{v}} + \int_{t_1}^{t_2} \sum \underline{\mathbf{F}}_{ext} dt = \sum m\underline{\mathbf{v}}^{*} \quad (1)$$

$$\sum m\underline{\mathbf{v}} = (5)(10\underline{\mathbf{i}}) + (10)(5\underline{\mathbf{i}}) = 100\underline{\mathbf{i}} \quad \frac{kg-m}{s}$$

$$\int_{t_1}^{t_2} \sum \underline{\mathbf{F}}_{ext} dt = 0 \quad \text{(No external forces)}$$

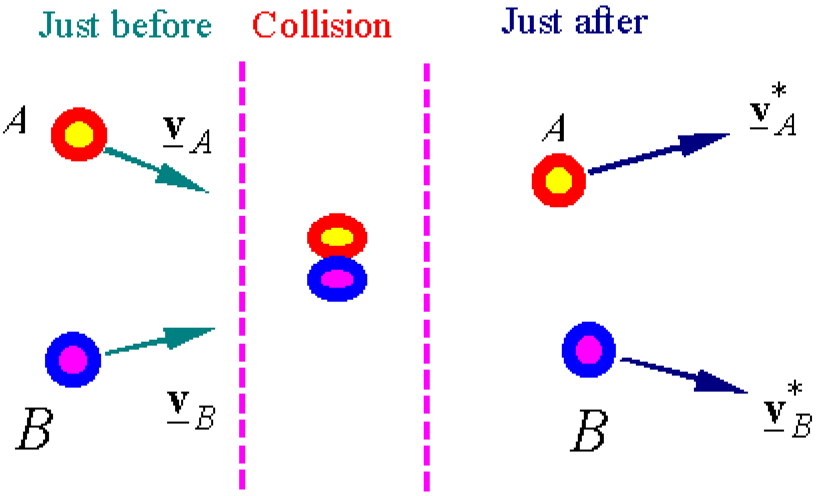
$$\sum m\underline{\mathbf{v}}^* = (5+10)v^*\underline{\mathbf{i}} \quad \text{(Both have the same velocity)}$$

Substitution into (1) gives

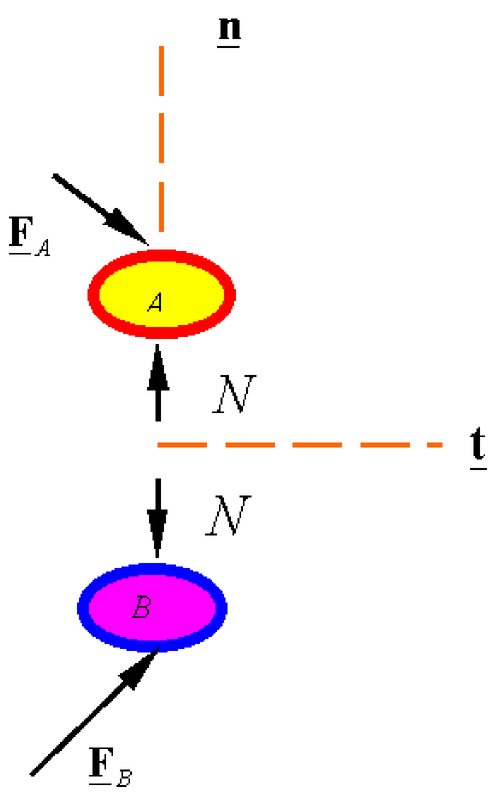
$$100\underline{\mathbf{i}} = 15v^*\underline{\mathbf{i}} \Rightarrow \quad v^* = 6.67 \quad \frac{m}{s}$$

ÇARPIŞMA

Frictionless impact: Impact refers to the interaction of two particles when the interaction interval is very short. Frictionless impact refers to short interactions when there is no noticeable effect due to friction.



The free-body diagram of the two particles during the collision is



The impulse momentum for each particle is

$$m_A \underline{\mathbf{v}}_A + \int_{t_1}^{t_2} (\underline{\mathbf{F}}_A + N\underline{\mathbf{n}})dt = m_A \underline{\mathbf{v}}_A^* \qquad (1)$$

$$m_B \underline{\mathbf{v}}_B + \int_{t_1}^{t_2} (\underline{\mathbf{F}}_A - N\underline{\mathbf{n}})dt = m_B \underline{\mathbf{v}}_B^* \qquad (2)$$

Where $\underline{\mathbf{F}}_A$ and $\underline{\mathbf{F}}_B$ are the externally applied forces. The general impulse momentum relation for this system will be

$$m_A \underline{\mathbf{v}}_A + m_B \underline{\mathbf{v}}_B + \int_{t_1}^{t_2} (\underline{\mathbf{F}}_A + \underline{\mathbf{F}}_B)dt = m_A \underline{\mathbf{v}}_A^* + m_B \underline{\mathbf{v}}_B^* \qquad (3)$$

An empirical relation is used to account for the loss of mechanical energy in the impact. This relation is between the relative speed that the particles approach each other and the relative speed that they separate from each other. This relation is

$$e = \frac{|(\underline{\mathbf{v}}_B^* - \underline{\mathbf{v}}_A^*) \circ \underline{\mathbf{n}}|}{|(\underline{\mathbf{v}}_B - \underline{\mathbf{v}}_A) \circ \underline{\mathbf{n}}|} \qquad (4)$$

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where e is the coefficient of restitution, an empirical constant between 0 and 1. The collision is considered elastic if $e=1$. In an elastic collision the magnitude of the normal component of the relative velocity before and after the collision is the same. The collision is considered fully plastic if $e=0$. In a plastic collision the normal component of the relative velocity of the two particles becomes zero (i.e., the component of the velocity of the two particles in the normal direction becomes the same).

Note that (3) is a linear combination of (1) and (2), and, therefore, is not an independent equation. Impact problems are solved using equation (4) and a combination of (1), (2), and (3).

When there is negligible impulse due to external forces: The time of the impact is assumed to be small. If $\underline{\mathbf{F}}_A$ and $\underline{\mathbf{F}}_B$ are bounded (must be less than infinity), when one lets $(t_2 - t_1)$ go to zero, the impulse of these forces go to zero and one can simplify the above set of equations to get

$$m_A(\underline{\mathbf{v}}_A)_n + m_B(\underline{\mathbf{v}}_B)_n = m_A(\underline{\mathbf{v}}_A^*)_n + m_B(\underline{\mathbf{v}}_B^*)_n \qquad (5)$$

$$\begin{cases} (\underline{\mathbf{v}}_A)_t = (\underline{\mathbf{v}}_A^*)_t & (6) \\ (\underline{\mathbf{v}}_B)_t = (\underline{\mathbf{v}}_B^*)_t & (7) \end{cases}$$

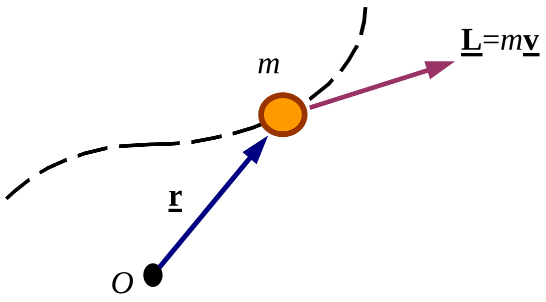
$$e = \frac{(\underline{\mathbf{v}}_B^*)_n - (\underline{\mathbf{v}}_A^*)_n}{(\underline{\mathbf{v}}_A)_n - (\underline{\mathbf{v}}_B)_n} \qquad (8)$$

The subscripts t and n refer to the components of the vectors along the $\underline{\mathbf{n}}$ and $\underline{\mathbf{t}}$ directions. Examples of bounded external forces are constant forces, such as the force of gravity, and forces resulting from motion, such as spring forces. In these cases, if the impact time is small, we can ignore the impulse of these forces and use equations (5)-(8).

The angular impulse-momentum relation

Angular momentum of a particle: The symbol $\underline{\mathbf{H}}_O$ denotes the angular momentum and is defined as the moment of linear momentum around the point *O*, and given by the equation

$$\underline{\mathbf{H}}_O = \underline{\mathbf{r}} \times \underline{\mathbf{L}}$$



ÖRNEK 1: THE ANGULAR IMPULSE-MOMENTUM RELATION

A particle of mass *m*=10 kg is at position (1,2,3) and has a velocity $\underline{\mathbf{v}} = 3\underline{\mathbf{i}} + 2\underline{\mathbf{j}} + \underline{\mathbf{k}}$ m/s. Calculate the angular momentum of this particle about point O with coordinates (0,2,1). The coordinates are given in meters.

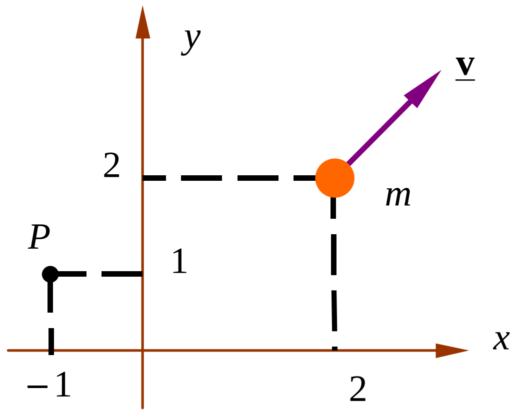
Çözüm:

$$\underline{\mathbf{r}} = (1-0)\underline{\mathbf{i}} + (2-2)\underline{\mathbf{j}} + (3-1)\underline{\mathbf{k}}$$
$$= \underline{\mathbf{i}} + 2\underline{\mathbf{k}} \;\; m$$

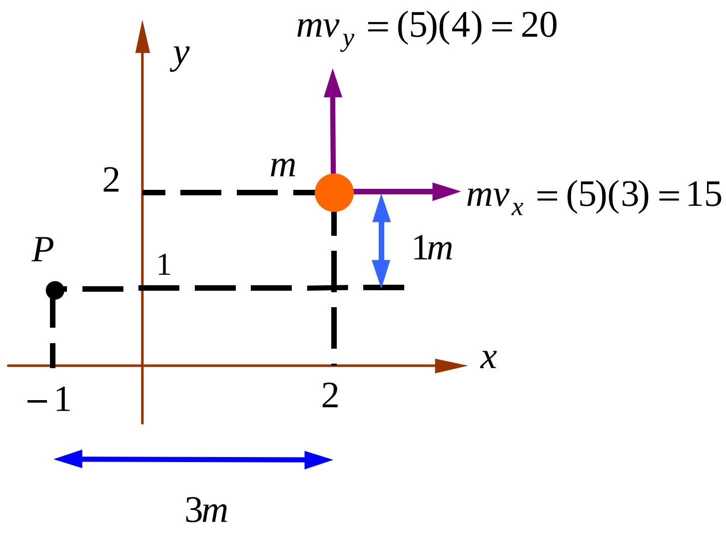
$$\underline{\mathbf{L}} \equiv m\underline{\mathbf{v}} = 10(3\underline{\mathbf{i}} + 2\underline{\mathbf{j}} + \underline{\mathbf{k}}) = 30\underline{\mathbf{i}} + 20\underline{\mathbf{j}} + 10\underline{\mathbf{k}} \quad \frac{kg-m}{s}$$

$$\underline{\mathbf{H}}_o \equiv \underline{\mathbf{r}} \times m\underline{\mathbf{v}} = (\underline{\mathbf{i}} + 2\underline{\mathbf{k}}) \times (30\underline{\mathbf{i}} + 20\underline{\mathbf{j}} + 10\underline{\mathbf{k}})$$
$$= 20\underline{\mathbf{k}} - 10\underline{\mathbf{j}} + (2)(30)\underline{\mathbf{j}} - (2)(20)\underline{\mathbf{i}}$$
$$= -40\underline{\mathbf{i}} + 50\underline{\mathbf{j}} + 20\underline{\mathbf{k}} \quad \frac{kg-m^2}{s}$$

ÖRNEK 2: THE ANGULAR IMPULSE-MOMENTUM RELATION



A particle of mass m=5 kg is at the position shown and has a velocity $\underline{\mathbf{v}} = 3\underline{\mathbf{i}} + 4\underline{\mathbf{j}}$ m/s. Calculate the angular momentum of this particle about point P.

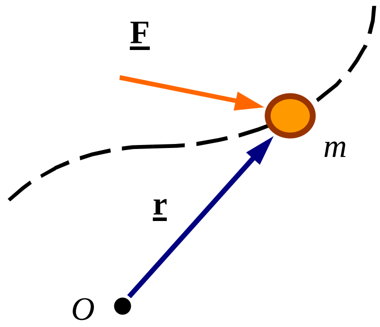


Solution:

Assuming counter clockwise to be positive

$$H_p = (3)(20) - (1)(15) = 45 kg - m^2/s$$

Impulse of a moment from time t_1 to t_2: The integral of the moment over the time interval of concern is its impulse. The impulse of a moment is a vector given by the integral



$$\underline{\mathbf{M}}_O = \underline{\mathbf{r}} \times \underline{\mathbf{F}}$$

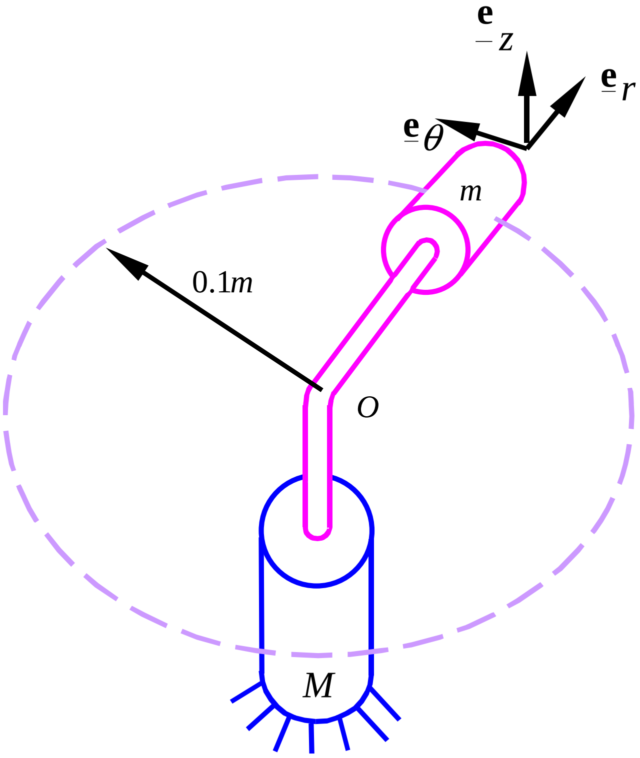
$$\int_{t_1}^{t_2} \underline{\mathbf{M}}_O dt$$

Moment and rate-of-change-of angular momentum relation: Starting from Newton's second law, calculating the moment of both sided of the equation with respect to a point *O* on an inertial frame results in

$$\begin{cases} \sum \underline{\mathbf{F}} = \dot{\underline{\mathbf{L}}} \Rightarrow \underline{\mathbf{r}} \times \sum \underline{\mathbf{F}} = \underline{\mathbf{r}} \times \dot{\underline{\mathbf{L}}} \\ \dot{\underline{\mathbf{H}}}_O = \underline{\mathbf{v}} \times \underline{\mathbf{L}} + \underline{\mathbf{r}} \times \dot{\underline{\mathbf{L}}} = \underline{\mathbf{r}} \times \dot{\underline{\mathbf{L}}} \Rightarrow \underline{\sum \underline{\mathbf{M}}_O = \dot{\underline{\mathbf{H}}}_O} \\ \underline{\mathbf{v}} \times \underline{\mathbf{L}} = \underline{\mathbf{v}} \times m\underline{\mathbf{v}} = 0 \end{cases}$$

This resulting relation between the moment and the rate of change of angular momentum is called the moment-rate of change of angular momentum relation (MRCAM).

ÖRNEK 3: THE ANGULAR IMPULSE-MOMENTUM RELATION



The motor M applies a constant moment M_o to rotate the particle. If the particle of mass m=20 kg starts from rest and the motor applies a moment M_o=10 N-m, calculate the speed of the particle after 10 seconds. Assume the particle is fixed to the bar at a radius of 0.1 m.

Solution:

$$\underline{\mathbf{H}}_{o_1} + \int_{t_1}^{t_2} \sum \underline{\mathbf{M}}_o dt = \underline{\mathbf{H}}_{o_2} \quad (1)$$

$$\underline{\mathbf{H}}_{o_1} = 0 \quad \text{(starts from rest, } \underline{\mathbf{v}} = 0)$$

$$\underline{\mathbf{H}}_{o_2} = \underline{\mathbf{r}} \times m\underline{\mathbf{v}} = (0.1\underline{\mathbf{e}}_r) \times (20)v\underline{\mathbf{e}}_\theta = 2v\underline{\mathbf{e}}_z$$

$$\int_{t_1}^{t_2} \sum \underline{\mathbf{M}}_o dt = \int_0^{10} M_o \underline{\mathbf{e}}_z dt = 10(10-0)\underline{\mathbf{e}}_z = 100\underline{\mathbf{e}}_z$$

substitution into (1) gives

$$100\underline{\mathbf{e}}_z = 2v\underline{\mathbf{e}}_z \Rightarrow v = 50 m/s$$

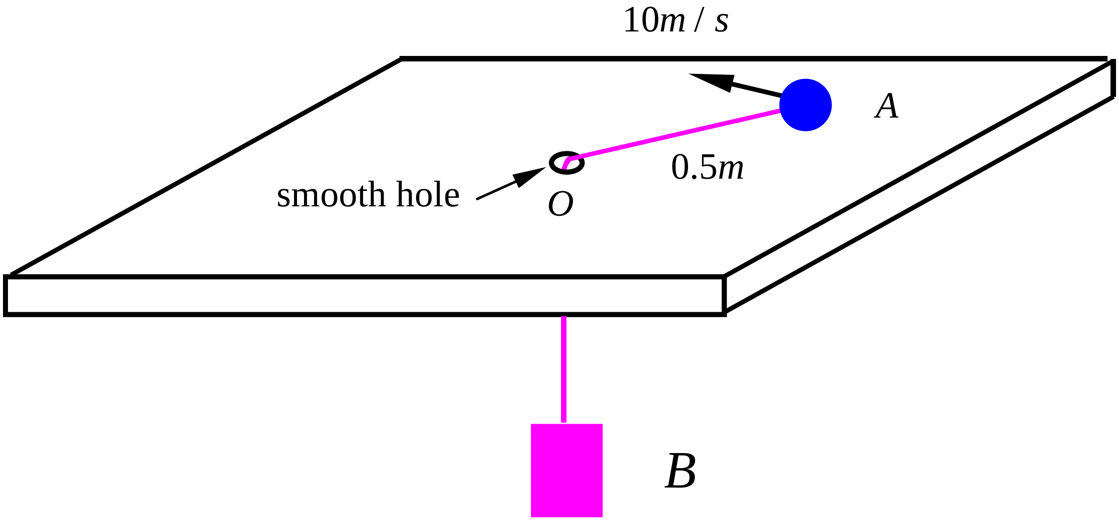
The angular impulse-momentum relation: Integration of Newton's 2nd law over the time interval from t_1 to t_2 results in

$$\int_{t_1}^{t_2} \sum \underline{\mathbf{M}}_O dt = \int_{t_1}^{t_2} \dot{\underline{\mathbf{H}}}_O dt \Rightarrow \int_{t_1}^{t_2} \sum \underline{\mathbf{M}}_O dt = \underline{\mathbf{H}}_O(t_2) - \underline{\mathbf{H}}_O(t_1) \Rightarrow$$

$$\boxed{\underline{\mathbf{H}}_{O_1} + \int_{t_1}^{t_2} \sum \underline{\mathbf{M}}_O dt = \underline{\mathbf{H}}_{O_2}}$$

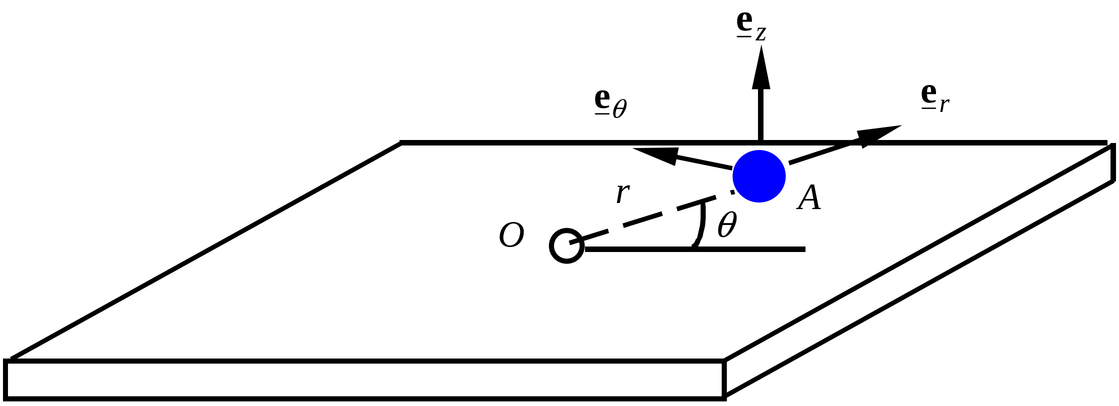
Therefore, the angular momentum of a particle is changed by the impulse of the resultant moment on the particle. There will be conservation of angular momentum only if the impulse of the resultant moment is zero.

ÖRNEK 4: THE ANGULAR IMPULSE-MOMENTUM RELATION

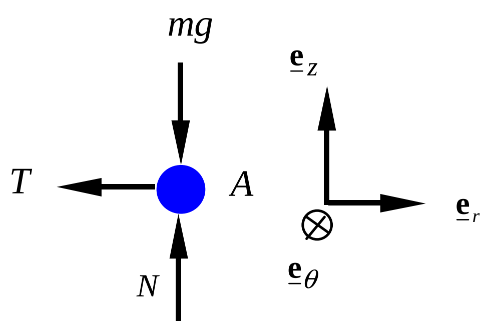


Particle *A* of mass 1 kg moves on a smooth horizontal surface and is connected by a string to particle *B*. *B* has a mass of 2 kg. Particle *B* is released from rest at time t=0 when particle *A* has a circumferential velocity of 10 m/s and a radius of 0.5 m. Write the equation of motion of the system.

Solution:



Particle A:



$$\sum F_z = ma_z \Rightarrow N - m_A g = 0 \quad (1)$$

$$\sum \underline{\mathbf{M}}_o = 0 \Rightarrow \underline{\mathbf{H}}_o = \text{constant}$$

$$\underline{\mathbf{H}}_o = \underline{\mathbf{r}} \times m\underline{\mathbf{v}} = (r\underline{\mathbf{e}}_r) \times m_A(\dot{r}\underline{\mathbf{e}}_r + r\dot{\theta}\underline{\mathbf{e}}_\theta) = m_A r^2 \dot{\theta}\underline{\mathbf{e}}_z$$

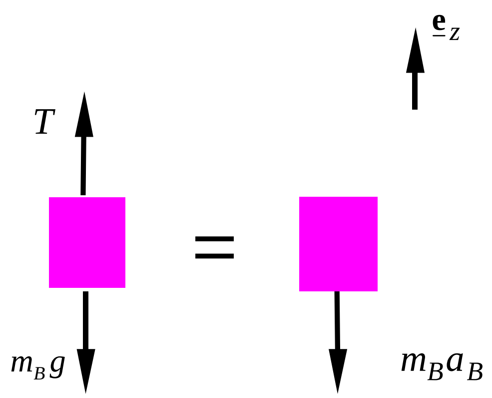
$$\underline{\mathbf{H}}_o = \text{constant} \Rightarrow r^2\dot{\theta} = r_o^2\dot{\theta}_o = (0.5)^2(\frac{10}{0.5}) = 5$$

$$\Rightarrow r^2\dot{\theta} = 5 \qquad (2)$$

$$\sum F_r = ma_r \Rightarrow -T = m_A(\ddot{r} - r\dot{\theta}^2)$$

$$\Rightarrow -T = \ddot{r} - r\dot{\theta}^2 \qquad (3)$$

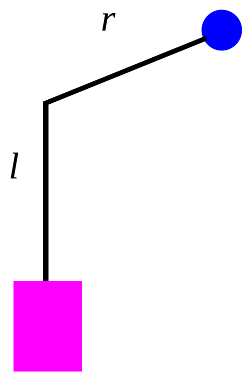
Particle B:



$$\sum F_z = ma_z \Rightarrow T - m_B g = -m_B a_B$$

$$\Rightarrow T - (2)(9.81) = -2a_B \qquad (4)$$

Kinematics relation:



$$r + l = \text{constant} \Rightarrow \ddot{r} = -\ddot{l}$$

$$\Rightarrow \ddot{r} = -a_B \quad (5)$$

Add (3) and (4) and substitute (5) to get

$$-19.62 = \ddot{r} - r\dot{\theta}^2 + 2\ddot{r}$$

Substitution from (2) gives

$$-19.62 = 3\ddot{r} - \frac{25}{r^3} \quad (6)$$

Equation (2) and (6) are the equation of motion for the system.