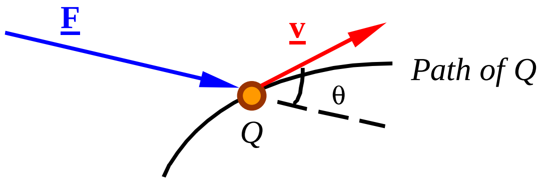
MÜHENDİSLİK MEKANİĞİ

11. HAFAT İş-Enerji

Power of a force: Power in the ability of a force to do work

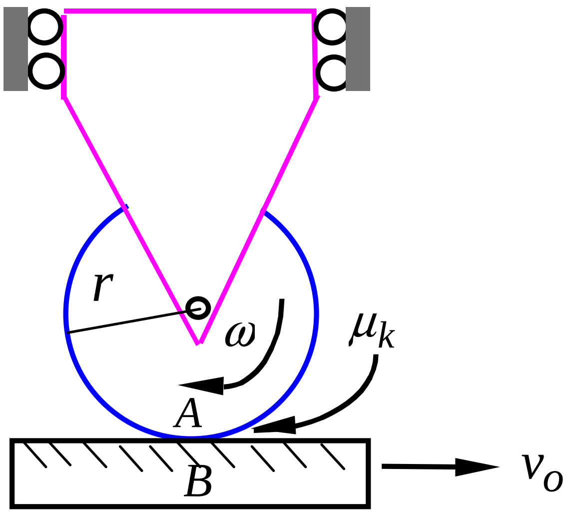
$$P = \mathbf{F} \circ \mathbf{v} = Fv\cos(\theta)$$

F: The force applied on particle *Q*



v: The velocity of *Q*

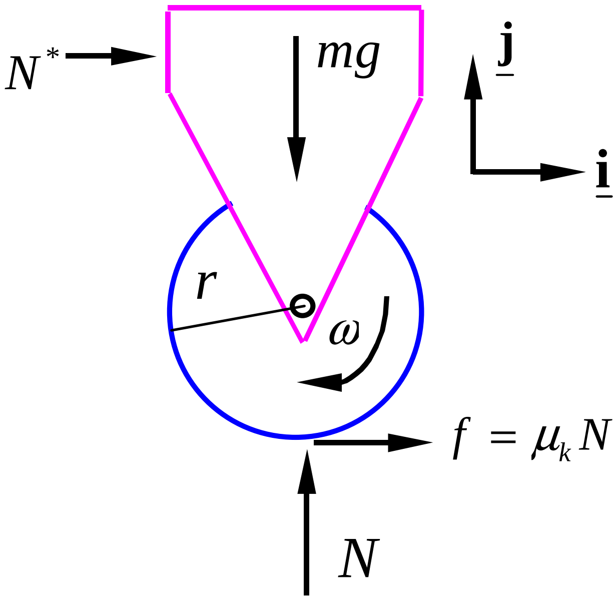
ÖRNEK1: İŞ-ENERJİ



Calculate the power of the frictional force to do work on the grinding disk and its power to do work on the work piece B. Assume the grinding wheel and connection have a total mass of *m*. What happens to the difference in the power?

Çözüm:

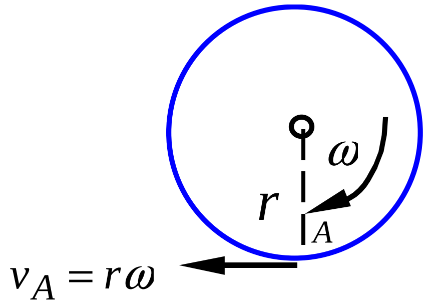
(a) Power of friction to do work on the grinding wheel



Kinetics:

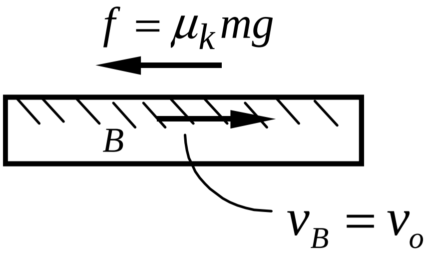
$\sum F_y = 0$

$$\Rightarrow N - mg = 0 \Rightarrow N = mg$$
$$\Rightarrow f = \mu_k mg$$



Kinematics:

$$P_A = \underline{\mathbf{F}} \circ \underline{\mathbf{v}} = (f\,\underline{\mathbf{i}}) \circ (-v_A \underline{\mathbf{i}})$$
$$= (\mu_k mg \underline{\mathbf{i}}) \circ (-r\omega\, \underline{\mathbf{i}}) = -\mu_k r\omega mg$$

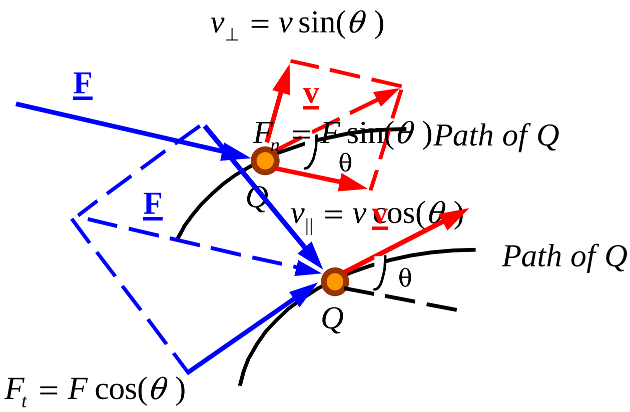


(b) Power of friction to do work on work piece *B*

$$P_B = \underline{\mathbf{F}} \circ \underline{\mathbf{v}} = (-f\,\underline{\mathbf{i}}) \circ (v_B \underline{\mathbf{i}}) = -fv_B = -\mu_k mgv_o$$

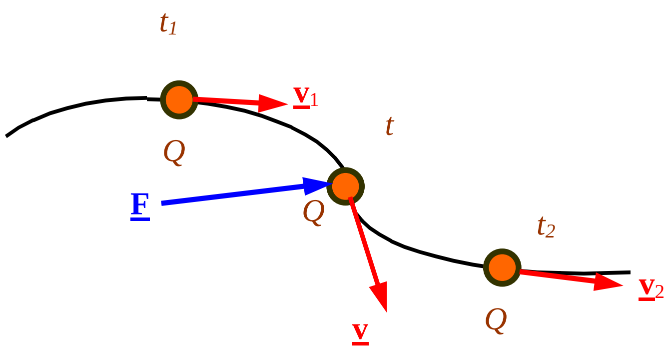
The difference in the power between P_A and P_B appears in the form of heat that ends up heating the surfaces.

Note: The power of a force is related to the velocity of the particle it is acting upon.



$$P = Fv\cos(\theta) = Fv_{||} = F_t v$$

Work of a force: The work $U_{1\text{-}2}$ of a force on a particle over the interval of time from t_1 to t_2 is the integral of its power over this time interval.



$$U_{1-2} = \int_{t_1}^{t_2} P dt = \int_{t_1}^{t_2} \underline{\mathbf{F}} \circ \underline{\mathbf{v}} dt = \int_{t_1}^{t_2} F_t v dt = \int_{t_1}^{t_2} F v_{||} dt$$

Other methods to find the work of a force are:

$$\left.\begin{array}{l} U_{1-2} = \int_{t_1}^{t_2} \underline{\mathbf{F}} \circ \underline{\mathbf{v}} dt \\ \underline{\mathbf{v}} = \dfrac{d\underline{\mathbf{r}}}{dt} \end{array}\right\} \Rightarrow U_{1-2} = \int_{t_1}^{t_2} \underline{\mathbf{F}} \circ \frac{d\underline{\mathbf{r}}}{dt} dt = \int_{\underline{\mathbf{r}}_1}^{\underline{\mathbf{r}}_2} \underline{\mathbf{F}} \circ d\underline{\mathbf{r}}$$

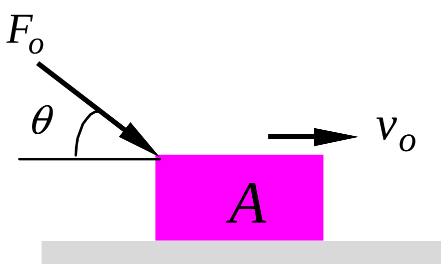
$$\Rightarrow U_{1-2} = \int_{\underline{\mathbf{r}}_1}^{\underline{\mathbf{r}}_2} (F_x \underline{\mathbf{i}} + F_y \underline{\mathbf{j}} + F_z \underline{\mathbf{k}}) \circ (dx \underline{\mathbf{i}} + dy \underline{\mathbf{j}} + dz \underline{\mathbf{k}})$$

$$\Rightarrow U_{1-2} = \int_{x_1}^{x_2} F_x dx + \int_{y_1}^{y_2} F_y dy + \int_{z_1}^{z_2} F_z dz$$

$$\left.\begin{array}{l} U_{1-2} = \int_{t_1}^{t_2} \underline{\mathbf{F}} \circ \underline{\mathbf{v}} dt \\ \underline{\mathbf{v}} = v \underline{e}_t \end{array}\right\} \Rightarrow U_{1-2} = \int_{t_1}^{t_2} \underline{\mathbf{F}} \circ v \underline{\mathbf{e}}_t dt = \int_{t_1}^{t_2} (\underline{\mathbf{F}} \circ \underline{\mathbf{e}}_t) v dt = \int_{t_1}^{t_2} F_t \frac{ds}{dt} dt$$

$$\Rightarrow U_{1-2} = \int_{s_1}^{s_2} F_t ds$$

ÖRNEK 2: İŞ- ENERJİ



Calculate the work from time 0 to time t of the constant magnitude force F_o that changes its direction by $\theta = \omega t$. Assume A moves with constant speed v_o.

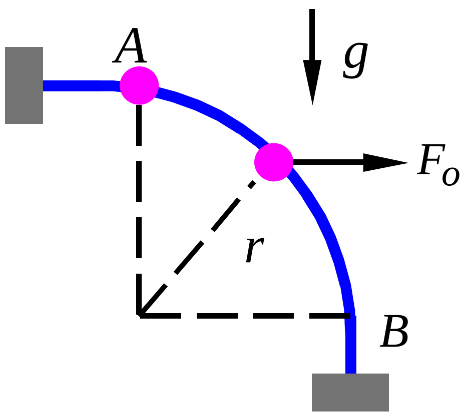
Çözüm:

$$U_{1-2} = \int_{t_1}^{t_2} \underline{\mathbf{F}} \circ \underline{\mathbf{v}} dt = \int_{t_1}^{t_2} F_t v dt$$

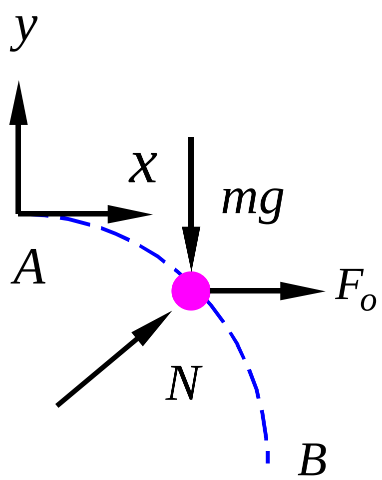
$$= \int_0^t F_o \cos\theta \, v_o dt = \int_0^t F_o \cos(\omega t) v_o dt$$

$$= \frac{F_o v_o}{\omega} \sin(\omega t)$$

ÖRNEK 3: İŞ - ENERJİ



A particle moves under the constant horizontal force F_o and gravity from point A to B on the smooth circular bar. Calculate the work of the forces applied on the particle.



Çözüm:

- Work of N: N is perpendicular to the path at all times and, as a result, has no component tangent to the path . Therefore, N does no work from A to B.
- • Work of F_o and *mg:*

$$U_{1-2} = \int_A^B \underline{\mathbf{F}} \circ d\underline{\mathbf{r}} = \int_A^B (F_o \underline{\mathbf{i}} - mg\underline{\mathbf{j}}) \circ (dx\underline{\mathbf{i}} + dy\underline{\mathbf{j}})$$

$$= \int_0^r F_o dx - \int_0^{-r} mgdy = F_o r + mgr$$

Kinetic energy of a particle: A particle of mass *m* at each instant in time has a kinetic energy *T* given by

$$\boxed{T = \frac{1}{2} m v^2}$$

The work-energy relation: The relation between the work done on a particle by the forces which are applied on it and how its kinetic energy changes follows from Newton's second law.

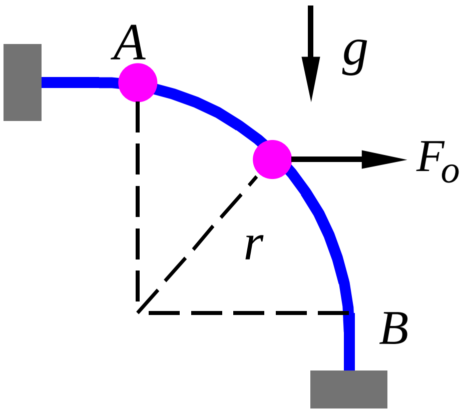
$$\sum \underline{\mathbf{F}} = m\underline{\mathbf{a}} \;\Rightarrow\; \sum \underline{\mathbf{F}} \circ \underline{\mathbf{v}} = m\underline{\mathbf{a}} \circ \underline{\mathbf{v}} \;\Rightarrow\; \sum \underline{\mathbf{F}} \circ \underline{\mathbf{v}} = m\underline{\mathbf{a}} \circ \underline{\mathbf{v}}$$

$$\left.\begin{array}{l} \displaystyle\int_{t_1}^{t_2} \sum \underline{\mathbf{F}} \circ \underline{\mathbf{v}} dt = \int_{t_1}^{t_2} m\underline{\mathbf{a}} \circ \underline{\mathbf{v}} dt \\ \displaystyle\frac{dv^2}{dt} = \frac{d\underline{\mathbf{v}} \circ \underline{\mathbf{v}}}{dt} = \underline{\mathbf{a}} \circ \underline{\mathbf{v}} + \underline{\mathbf{v}} \circ \underline{\mathbf{a}} = 2\underline{\mathbf{a}} \circ \underline{\mathbf{v}} \end{array}\right\} \Rightarrow \sum U_{1-2} = T_2 - T_1$$

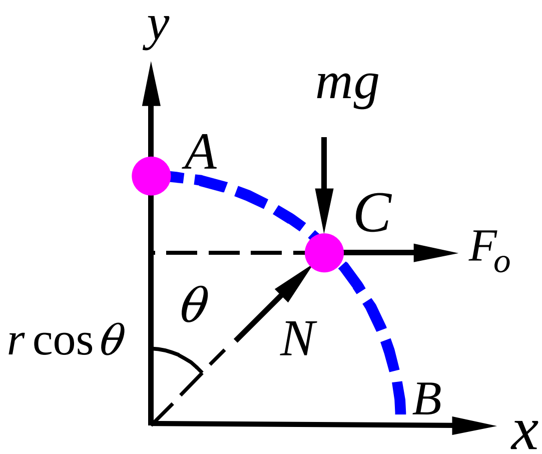
Note: The work done on a particle by the resultant force applied on it over a given interval of time will be equal to the change in kinetic energy of the particle. In other words, the kinetic energy of a particle is changed by an amount equal to the work which is done on the particle by the resultant force.

$$\boxed{T_1 + \sum U_{1-2} = T_2}$$

ÖRNEK 4: İŞ- ENERJİ



The particle starts from rest at A and moves along the smooth circular bar under the constant horizontal force F_o and under gravity. Calculate the velocity of the particle as a function of θ .



Solution:

- Work of N equals zero since it is perpendicular to the path at all the times.
- Work of F_o and mg

1. $$\sum U_{A-C} = \int_A^C \underline{\mathbf{F}} \circ d\underline{\mathbf{r}} = \int_A^C (F_o\underline{\mathbf{i}} - mg\underline{\mathbf{j}}) \circ (dx\underline{\mathbf{i}} + dy\underline{\mathbf{j}})$$

2. $$= \int_0^{r\sin\theta} F_o dx - \int_r^{r\cos\theta} mgdy = F_o r\sin\theta - mgr(\cos\theta - 1)$$

3. $$\Rightarrow \sum U_{A-C} = F_o r \sin\theta + mgr(1-\cos\theta) \quad (1)$$

- • Work-Energy relation:

$$T_A + \sum U_{A-C} = T_C \quad (2)$$

$$T_A = \frac{1}{2} m v_A^2 = 0, \quad (v_A = 0, \text{ start from rest})$$

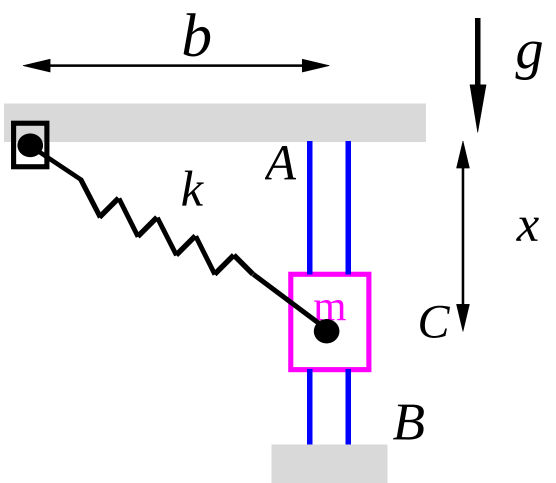
$$T_C = \frac{1}{2} m v_C^2$$

(1) (1) into (2)

$$\Rightarrow F_o r \sin\theta + mgr(1-\cos\theta) = \frac{1}{2} m v_C^2$$

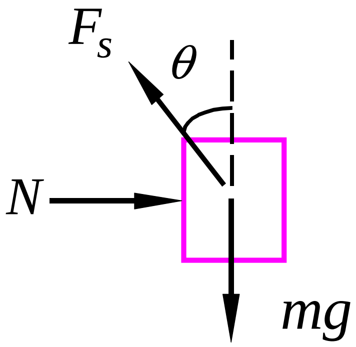
$$\Rightarrow v_C = \sqrt{2\frac{F_o r}{m}\sin\theta + 2gr(1-\cos\theta)}$$

ÖRNEK 5: İŞ - ENERJİ



The particle starts from rest at A. If the bar is smooth and the spring has a free length of l_o, calculate the speed of the particle as a function of *x*.

Solution:



- • First calculate the work of the forces.
- • N does no work since it is perpendicular to the path of the bar at all times.

$$\sum U_{A-C} = \int_A^B F_t ds = \int_0^x (-F_s \cos\theta + mg)dx$$

$$= \int_0^x [-k(\sqrt{x^2+b^2} - l_o)\frac{x}{\sqrt{x^2+b^2}} + mg]dx$$

$$= \int_0^x [-kx + kl_o \frac{x}{\sqrt{x^2+b^2}} + mg]dx$$

$$= -k\frac{x^2}{2} + kl_o\sqrt{x^2+b^2} - kl_o b + mgx$$

Start from rest $\Rightarrow v_A = 0$

$$T_A = \frac{1}{2}mv_A^2 = 0$$

$$T_C = \frac{1}{2}mv_B^2 = \frac{1}{2}m\dot{x}^2$$

Work-Energy relation

$$T_A + \sum U_{A-B} = T_B$$

$$\Rightarrow 0 - \frac{kx^2}{2} + kl_o\sqrt{x^2+b^2} - kl_o b + mgx = \frac{1}{2}m\dot{x}^2$$

$$\Rightarrow \dot{x} = \sqrt{-\frac{kx^2}{m} + \frac{2}{m} kl_o \sqrt{x^2 + b^2} - \frac{2}{m} kl_o b + 2gx}$$

POTANSİYEL ENERJİ

Conservative forces: A force $\underline{\mathbf{F}}$ is conservative if it can be represented as the gradient of a scalar function. This can written as

$$\underline{\mathbf{F}} = -\underline{\nabla} V$$

where V is a scalar function of position in space. The negative sign is only for convenience so that V can be identified as the potential energy.

Work of a conservative force: Given a conservative force $\underline{\mathbf{F}}$, its work can be calculated as

$$U_{1-2} = \int_{t_1}^{t_2} \underline{\mathbf{F}} \circ \underline{\mathbf{v}} dt = \int_{t_1}^{t_2} -\underline{\nabla} V \circ \underline{\mathbf{v}} dt$$

$$= -\int_{t_1}^{t_2} \left(\frac{\partial V}{\partial x}\underline{\mathbf{i}} + \frac{\partial V}{\partial y}\underline{\mathbf{j}} + \frac{\partial V}{\partial z}\underline{\mathbf{k}}\right) \circ \left(\frac{dx}{dt}\underline{\mathbf{i}} + \frac{dy}{dt}\underline{\mathbf{j}} + \frac{dz}{dt}\underline{\mathbf{k}}\right) dt$$

$$= -\int_{t_1}^{t_2} \left(\frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt}\right) dt = -\int_{t_1}^{t_2} \frac{dV}{dt} dt$$

$$= -[V(t_2) - V(t_1)]$$

Therefore, the work of a conservative force is negative the change in its potential energy.

Note: Since the work of a conservative force does not depend on the path the particle takes when going from its location at time 1 to its location at time 2, the work of a conservative force is path independent.

ÖRNEK 1: POTANSİYEL ENERJİ

Calculate the potential energy of a constant force $\underline{\mathbf{F}}$.

Solution:

$$\underline{\mathbf{F}} = -\nabla V \Rightarrow \begin{cases} F_x = -\dfrac{\partial V}{\partial x} & (1) \\ F_y = -\dfrac{\partial V}{\partial y} & (2) \\ F_z = -\dfrac{\partial V}{\partial z} & (3) \end{cases}$$

Since F̲ is constant, then F_x, F_y, F_z are constant. V is a function of (*x*,*y*,*z*). Therefore

$$(1) \Rightarrow V = -F_x x + C(y, z) \quad (4)$$

Note that the constant of integration is not a real constant but is a function of *y* and *z*.

Substitution of (4) into (2) GIVES

$$F_y = -\frac{\partial V}{\partial y} = -\frac{\partial C}{\partial y} \Rightarrow C = -F_y y + D(z) \quad (5)$$

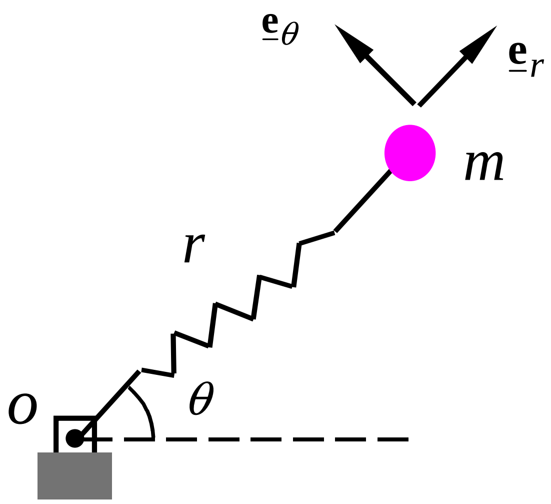
Substitution of (5) into (3) gives

$$F_z = -\frac{\partial V}{\partial z} = -\frac{\partial D}{\partial z} \Rightarrow D = -F_z z + E$$

Therefore, the potential energy of a constant force **F̲** is

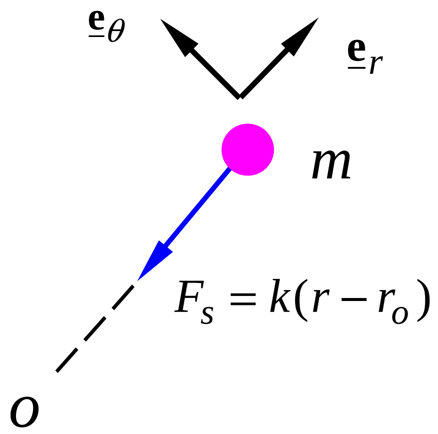
$$V = -F_x x - F_y y - F_z z + E$$

ÖRNEK 2: POTANSİYEL ENERJİ



Calculate the potential energy of the spring force. Assume the spring has a free length of r_o.

Solution for 2-D motion



In polar coordinates

$$\underline{\nabla} V = \frac{\partial V}{\partial r}\underline{\mathbf{e}}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\underline{\mathbf{e}}_\theta$$

Therefore,

$$\underline{\mathbf{F}}_s = \underline{\nabla} V \Rightarrow \begin{cases} -k(r - r_o) = -\dfrac{\partial V}{\partial r} & (1) \\ 0 = -\dfrac{1}{r}\dfrac{\partial V}{\partial \theta} & (2) \end{cases}$$

Integrating (2) gives

$$V = C(r) \quad (3)$$

Substituting (3) into (4) gives

$$-k(r - r_o) = -\frac{dC}{dr} \Rightarrow k(r - r_o)dr = dC$$

Integration using $w = r - r_o$ so that $dw = dr$ gives

$$k\frac{(r - r_o)^2}{2} + D = C$$

Therefore, the potential energy for the spring force is

$$V = \frac{1}{2}k(r - r_o)^2 + D$$

The work-energy relation: The work energy relation can be rewritten to take advantage of the fact that you can calculate the work of conservative forces form the potential energy. Let the resultant work of all forces on a particle be separated into the work done by conservative forces and the work done by non-conservative forces as follows

$$\sum U_{1-2} = \sum U_{1-2\,Con.} + \sum U_{1-2\,non-con} = -\sum(V_2 - V_1) + \sum U_{1-2\,non-con}$$

Introduction of this into the work energy relation gives

$$T_1 + \sum U_{1-2} = T_2 \quad \Rightarrow T_1 - \sum(V_2 - V_1) + \sum U_{1-2\,non-con} = T_2$$

$$\Rightarrow \quad T_1 + \sum V_1 + \sum U_{1-2\,non-con} = T_2 + \sum V_2$$

Total Mechanical Energy: The total mechanical energy E is the sum of the kinetic energy and the potential energy of the conservative forces.

$$E = T + \sum V$$

The work energy relation can now be written as

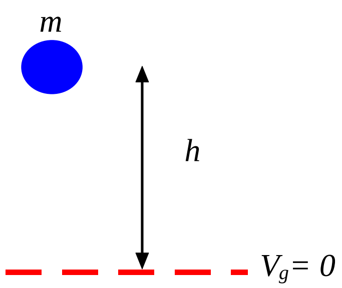
$$E_1 + \sum U_{1-2\,non-cons} = E_2$$

Conservative systems: A conservative system is one for which the total mechanical energy E remains constant. This can only happen if the non-conservative forces do no work.

Potential energy of gravity: The potential energy of gravity is given by

$$V_g = mgh$$

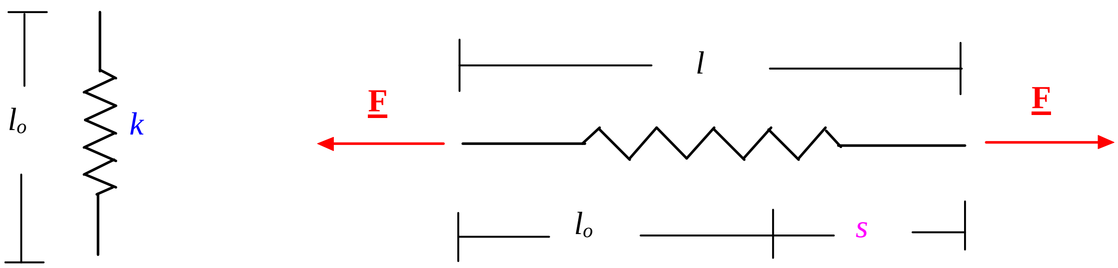
where h is measured from an elevation selected as reference.



Potential energy of a spring: The potential energy of a spring is given by

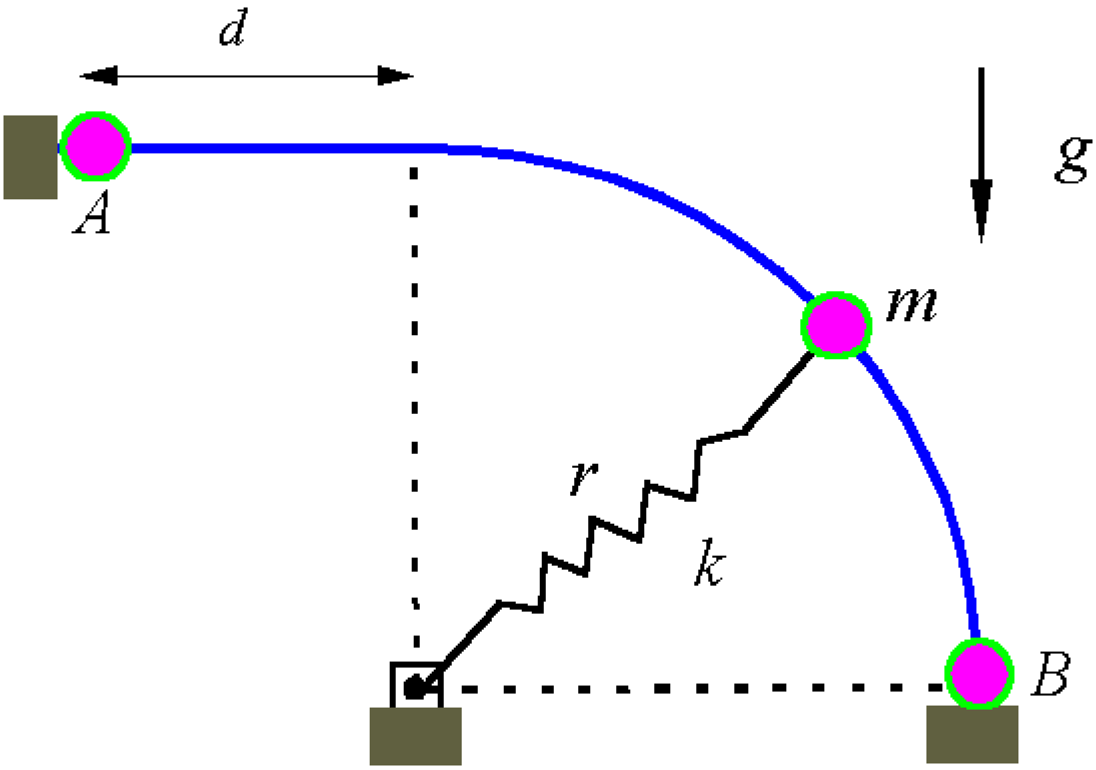
$$V_s = \frac{1}{2}k(l - l_o)^2 = \frac{1}{2}ks^2$$

where k is the spring stiffness, l is the spring's current length, and l_o is the free length of the spring, and s is the stretch of the spring.



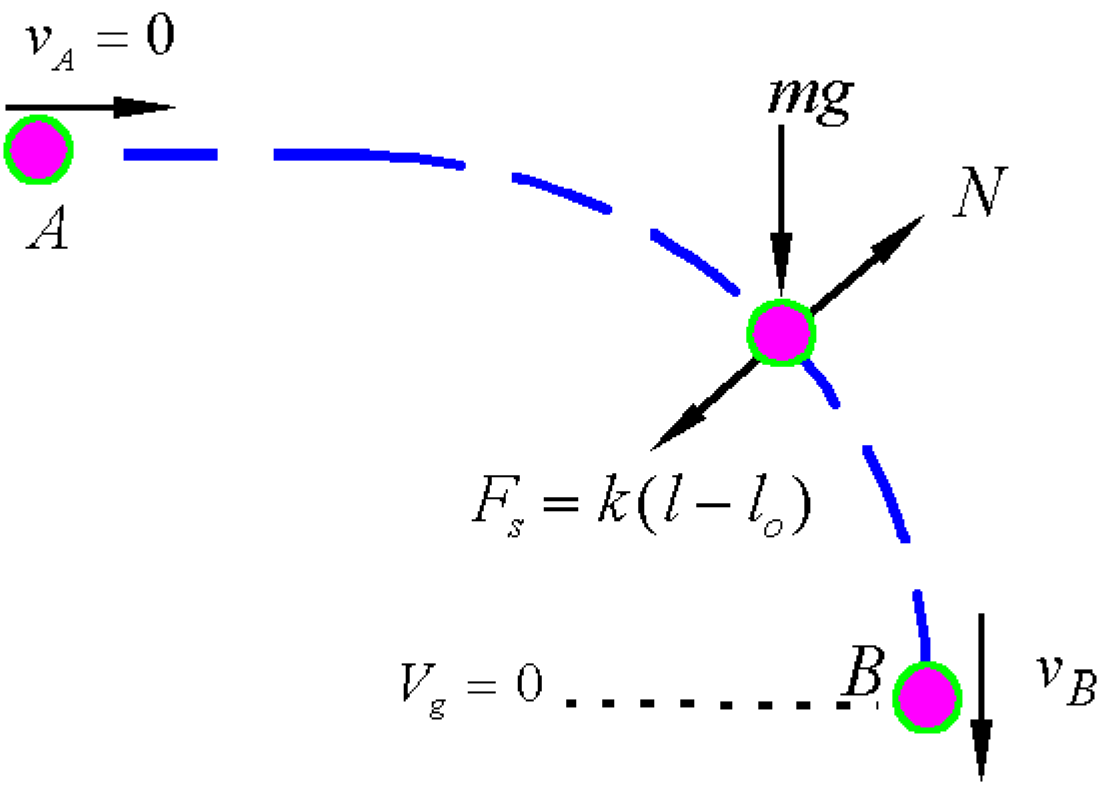
Free Spring Stretched Spring

ÖRNEK 3: POTANSİYEL ENERJİ



Mass m is released from rest at A and moves on the smooth collar from A to B under the action of the spring and gravity. Calculate the speed of the particle before it hits B. The spring has an initial length of $l_o = \dfrac{r}{2}$.

Solution:



Force analysis:

- *mg* is conservative
- F_s is conservative
- *N* is perpendicular to the path at all points from A to B, so it does no work.

Balance of work and energy:

$$T_A + \sum V_A + \sum U_{A-B\ \mathrm{non-con}} = T_B + \sum V_B \quad (1)$$

Starts from rest at A $\Rightarrow v_A = 0$

$$T_A = \frac{1}{2} m v_A^2 = 0$$

$$T_B = \frac{1}{2} m v_B^2$$

$$\sum V_A = V_{gA} + V_{sA} = mgh_A + \frac{1}{2}k(l_A - l_o)^2$$

$$= mgr + \frac{1}{2}k(\sqrt{d^2 + r^2} - \frac{r}{2})^2$$

$$\sum V_B = V_{gB} + V_{sB} = mgh_B + \frac{1}{2}k(l_B - l_o)^2$$

$$= mg(0) + \frac{1}{2}k(r - \frac{r}{2})^2 = \frac{1}{8}kr^2$$

$\sum U_{A-B\ \text{non-con}} = 0$ (all forces which do work are conservative)

Substitution into (1) gives

$$mgr + \frac{1}{2}k(\sqrt{d^2 + r^2} - \frac{r}{2})^2 = \frac{1}{2}mv_B^2 + \frac{1}{8}kr^2$$

$$\Rightarrow v_B = \sqrt{2gr + \frac{k}{m}(\sqrt{d^2 + r^2} - \frac{r}{2})^2 - \frac{k}{4m}r^2}$$