Impedance of a Coil Surrounding an Infinite Cylinder with an Arbitrary Radial Conductivity Profile

Erol Uzal, Ibrahim Ozkol, and Metin O. Kaya

Abstract— **A numerical-analytical method of computing the impedance of a cylindrical coil surrounding a cylindrical work-piece with an arbitrary radial conductivity change is presented. The coil is driven by a constant current source at a fixed frequency. Maxwell's equations are used in the quasi-static approximation and the work-piece is assumed to be nonmagnetic. The solution can be used in the modeling of eddy-current nondestructive evaluation of cylindrical parts.**

Index Terms— **Coil impedance, eddy-current, nondestructive evaluation.**

I. Introduction

INFERRING one-dimensional (1-D) property variations is an important problem in eddy-current nondestructive evaluation (NDE). Its applications include estimating the thickness, conductivity, and approximate profiles of surface layers resulting from processes like cladding, nitriding, heat treating, and ion implantation. This is an example of an inverse problem in which some measured terminal quantity, like voltage or impedance, is used to estimate the properties of the workpiece. The forward solution of the problem, i.e., the solution for the impedance or voltage of the eddy-current probe coil when the layer properties are known, is used to compute the layer properties by backfitting the measured data.

The works of Cheng [1], Dodd and Deeds [2], Cheng *et al.* [3], and Uzal *et al.* [4] provide the necessary forward solutions in analytical form for plane and cylindrical bodies. Norton and Kahn [5], Moulder *et al.* [6], and Uzal *et al.* [7] used exact forward solutions and least square minimization to solve the inverse problem. In this study, we would like to propose a numerical method for computing the impedance of a right cylindrical coil encircling an infinite cylindrical workpiece that has an arbitrary radial conductivity profile. Properties of the cylinder do not change in the axial direction. The method is based on an analytical solution of the problem with the same geometry, but with a conductivity profile that is piecewise constant. We take the magnetic permeability as constant and equal to the magnetic permeability of free space, but it could easily be included as a piecewise constant function in the solution.

In Section II, we develop the solution for a piecewise constant conductivity profile and give an expression for the computation of the impedance change from an unprocessed cylinder, i.e., the impedance of the coil for a cylinder with constant conductivity minus the impedance for the coil with piecewise constant radial-conductivity profile. In Section III, after introducing the numerical method and testing its convergence, we give numerical results for several conductivity profiles. Finally, we conclude the paper with a discussion in Section IV.

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II. Analytical Solution for a Piecewise Constant Conductivity Profile

In this section we develop the analytical solution for a cylinder with a piecewise constant radial-conductivity profile. First, the solution for a single-turn coil is considered and later the solution for an n-turn coil is formed by superposition.

The common axis of the cylinder and the coil is denoted as the z-axis (Fig. 1). We use the cylindrical coordinate system. The coil's radius is r_0 and it is located at the axial position z_0. The cylinder consists of various layers as shown in Fig. 1. The interface between layers k and $k+1$ is located at r_k. There are a total of $N-2$ layers of material including the cylinder core. The conductivity of layer k is σ_k. Layers $N-1$ and N are air with zero conductivity. The coil is located at the interface of these two layers. The magnetic permeability is μ_0 (magnetic permeability of free space) everywhere. The coil is driven by a constant current source of angular frequency ω and magnitude I. Therefore, any field quantity $F(\vec{r}, t)$ will have the time dependence as

$$F(\vec{r}, t) = \mathrm{Re}\{F(\vec{r})e^{i\varpi t}\} \tag{1}$$

where $\vec{r}$ denotes the space coordinates. In eddy-current NDE, ω is low enough for the quasi-static approximation to be made. Furthermore, using the vector potential with the Coulomb gauge, based on the assertion that no free charges exist, the equation that the vector potential satisfies can be written as

$$\nabla^2 \vec{A} - i\varpi\mu_0\sigma(r)\vec{A} = 0 \tag{2}$$

where $\sigma(r)$ is the piecewise constant conductivity function. Electric and magnetic field intensities are related to the vector potential by

$$\vec{E} = -i\varpi\vec{A}, \qquad \vec{H} = -\frac{1}{\mu_0}\nabla\times\vec{A}.$$

Since the problem is axially symmetrical, the vector potential has only an azimuthal component that depends on r and z

$$\vec{A} = A(r, z)\vec{e}_\theta \tag{3}$$

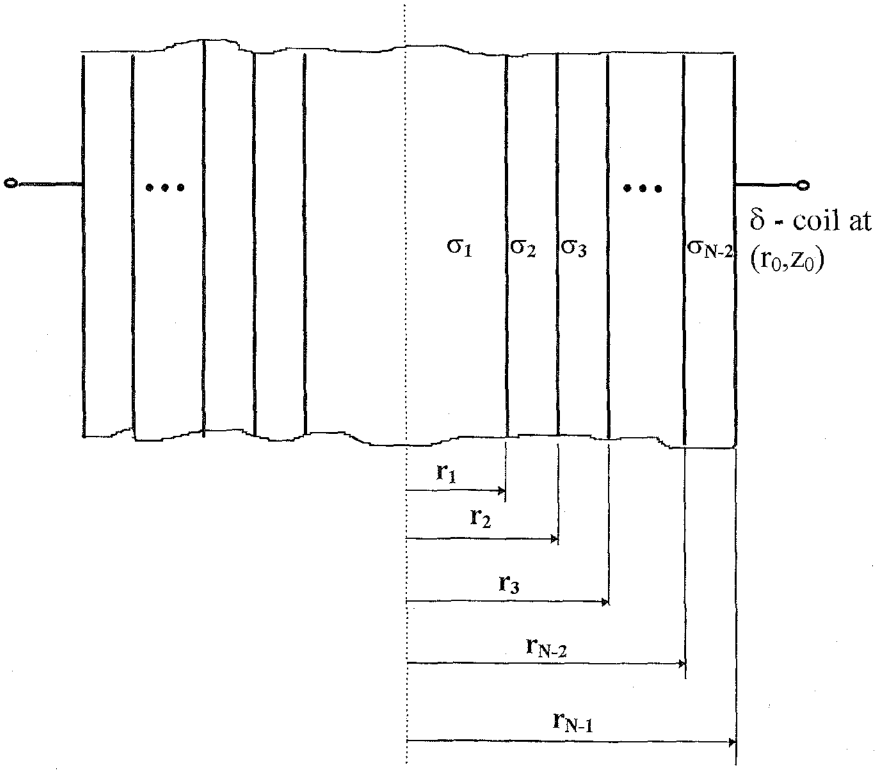

Fig. 1. Multilayered cylindrical workpiece surrounded by a single-turn coil.

where $\vec{e}_\theta$ is the unit vector in circumferential direction. Therefore (2) reduces to the scalar equation

$$\frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r} - \frac{A}{r^2} + \frac{\partial^2 A}{\partial z^2} = i\varpi\mu_0\sigma(r)A. \tag{4}$$

We denote the vector potential in region k by $A^{(k)}(r, z)$, where $\sigma(r)$ is equal to the constant σ_k. Thus, $A^{(k)}(r, z)$ satisfies

$$\frac{\partial^2 A^{(k)}}{\partial r^2} + \frac{1}{r}\frac{\partial A^{(k)}}{\partial r} - \frac{A^{(k)}}{r^2} + \frac{\partial^2 A^{(k)}}{\partial z^2} = i\varpi\mu_0\sigma_k A^{(k)}. \tag{5}$$

We find, by using separation of variables, that the solutions to (5) should be of the form

$$A^{(k)} = \int_0^\infty [B_k(\alpha)I_1(\alpha_k r) + C_k(\alpha)K_1(\alpha_k r)] \cdot \cos\alpha(z - z_0)\, d\alpha. \tag{6}$$

The form of (6) assures that the solution is symmetrical about $z = z_0$. I_1 and K_1 are the modified Bessel functions (or the Bessel functions of imaginary argument). α_k is defined as

$$\alpha_k = \sqrt{\alpha^2 + i\varpi\mu_0\sigma_k}, \qquad k = 1, 2, \cdots, N. \tag{7}$$

$B_k(\alpha)$ and $C_k(\alpha)$ are found from jump and boundary conditions. At the interface between two layers (regions), the jump conditions are that the tangential electric field should be continuous and the jump in tangential magnetic field should be equal to the surface current at the interface. These lead to

$$A^{(m)}(r_m, z) = A^{(m+1)}(r_m, z) \tag{8a}$$

$$\frac{\partial A^{(m)}}{\partial r}(r_m, z) = \frac{\partial A^{(m+1)}}{\partial r}(r_{m+1}, z) \tag{8b}$$

for $m = 1, 2, \cdots, N-2$, and for $m = N-1$ we have

$$A^{(N-1)}(R_0, z) = A^{(N)}(R_0, z) \tag{8c}$$

$$\frac{\partial A^{(N-1)}}{\partial r}(R_0, z) = \frac{\partial A^{(N)}}{\partial r}(R_0, z) + \mu_0 I\delta(z - z_0) \tag{8d}$$

where δ is the Dirac delta distribution. There are no surface currents except at the coil's position. We must add to these the conditions

$$A \text{ is finite at } r = 0 \tag{9a}$$

$$A \to 0 \text{ as } r \to \infty \tag{9b}$$

which can be satisfied by taking

$$C_1 = 0 \quad \text{and} \quad B_N = 0.$$

Using (6) in (8) and applying the Fourier cosine theorem, we obtain the equations for the unknown coefficients

$$\begin{bmatrix} K_1(\alpha_m r_m) & I_1(\alpha_m r_m) \\ K_1'(\alpha_m r_m) & I_1'(\alpha_m r_m) \end{bmatrix} \cdot \begin{bmatrix} B_m \\ C_m \end{bmatrix} = \begin{bmatrix} K_1(\alpha_{m+1} r_m) & I_1(\alpha_{m+1} r_m) \\ \frac{\alpha_{m+1}}{\alpha_m} K_1'(\alpha_{m+1} r_m) & \frac{\alpha_{m+1}}{\alpha_m} I_1'(\alpha_{m+1} r_m) \end{bmatrix} \cdot \begin{bmatrix} B_{m+1} \\ C_{m+1} \end{bmatrix} \tag{10}$$

for $m = 1, 2, \cdots, N-2$, and for $m = N-1$ we have

$$\begin{bmatrix} K_1(\alpha_{N-1} r_{N-1}) & I_1(\alpha_{N-1} r_{N-1}) \\ K_1'(\alpha_{N-1} r_{N-1}) & I_1'(\alpha_{N-1} r_{N-1}) \end{bmatrix} \cdot \begin{bmatrix} B_{N-1} \\ C_{N-1} \end{bmatrix} = \begin{bmatrix} K_1(\alpha_N r_{N-1}) \\ K_1'(\alpha_N r_{N-1}) \end{bmatrix} B_N + \begin{bmatrix} 0 \\ \frac{\mu_0 I}{\pi\alpha} \end{bmatrix}. \tag{11}$$

We rewrite (10) as follows:

$$\begin{bmatrix} B_{m+1} \\ C_{m+1} \end{bmatrix} = H_m \begin{bmatrix} B_m \\ C_m \end{bmatrix}, \qquad m = 1, 2, \cdots, N-2 \tag{12}$$

where

$$H_m = r_m \begin{bmatrix} a_{m11} & a_{m12} \\ a_{m21} & a_{m22} \end{bmatrix} \tag{13}$$

with

$$\begin{aligned} a_{m11} &= \alpha_{m+1} I_0(\alpha_{m+1} r_m) K_1(\alpha_m r_m) \\ &\quad + \alpha_m I_1(\alpha_{m+1} r_m) K_0(\alpha_m r_m) \end{aligned} \tag{14a}$$

$$\begin{aligned} a_{m12} &= \alpha_{m+1} I_0(\alpha_{m+1} r_m) I_1(\alpha_m r_m) \\ &\quad - \alpha_m I_1(\alpha_{m+1} r_m) I_0(\alpha_m r_m) \end{aligned} \tag{14b}$$

$$\begin{aligned} a_{m21} &= \alpha_{m+1} K_0(\alpha_{m+1} r_m) K_1(\alpha_m r_m) \\ &\quad - \alpha_m K_1(\alpha_{m+1} r_m) K_0(\alpha_m r_m) \end{aligned} \tag{14c}$$

$$\begin{aligned} a_{m22} &= \alpha_{m+1} K_0(\alpha_{m+1} r_m) I_1(\alpha_m r_m) \\ &\quad + \alpha_m K_1(\alpha_{m+1} r_m) I_0(\alpha_m r_m). \end{aligned} \tag{14d}$$

To obtain an explicit expression for the impedance, we need to evaluate B_{N-1}, C_{N-1}, and B_N. For this purpose, using (12) iteratively, we obtain

$$\begin{bmatrix} B_{N-1} \\ C_{N-1} \end{bmatrix} = H \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \tag{15}$$

where H is the product of 2×2 matrices

$$H = H_{N-2} H_{N-3} \cdots H_2 H_1 = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}. \tag{16}$$

Thus

$$\frac{B_{N-1}}{C_{N-1}} = \frac{h_{11}}{h_{21}} = \phi(\alpha, r_1, \cdots, r_{N-2}, \sigma_1, \cdots, \sigma_{N-2}). \tag{17}$$

Solving the system consisting of (11) and (17) for B_{N-1}, C_{N-1}, and B_N, we obtain

$$A = \frac{\mu_0 I}{\pi} r_0 \int_0^\infty Q(\alpha, r, r_0) \cos \alpha(z - z_0)\, d\alpha, \qquad r > r_{N-2} \tag{18a}$$

where

$$Q(\alpha, r, r_0) = \begin{cases} [\phi(\alpha) K_1(\alpha r) + I_1(\alpha r)] K_1(\alpha r_0) & r < r_0 \\ [\phi(\alpha) K_1(\alpha r_0) + I_1(\alpha r_0)] K_1(\alpha r) & r > r_0 \end{cases}. \tag{18b}$$

Now, we consider an n-turn coil of rectangular cross section encircling the cylinder (Fig. 2). Vector potential for this case is written by superposition as

$$A_{n\text{-turn}} = \int_{R_1}^{R_2} \int_{L_1}^{L_2} A(r, z, r_0, z_0)\, dz_0\, dr_0 \tag{19}$$

where $A(r, z, r_0, z_0)$ is the solution for the single-turn coil problem given by (18), except that I should be replaced by the current density $J(r_0, z_0)$. We will assume that the current is uniformly distributed over the rectangular cross section of the coil, which means J is constant

$$J = \frac{nI}{(L_2 - L_1)(R_2 - R_1)}. \tag{20}$$

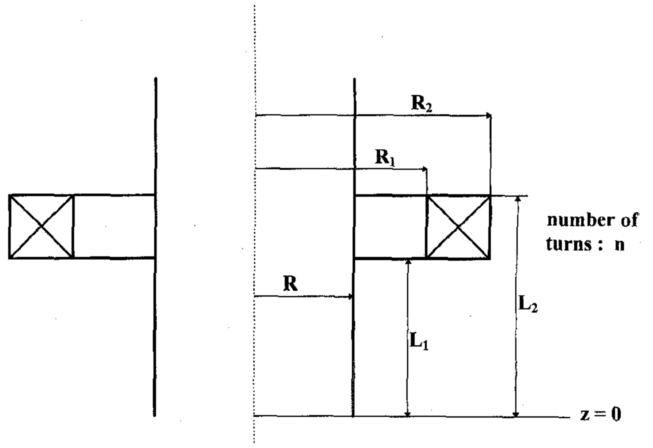

Fig. 2. Geometry of the problem with an n-turn coil. $R = r_{N-2}$ is the radius of the workpiece.

To be able to compute the coil impedance, we should write $A_{n\text{-turn}}$ in the region that the coil occupies, i.e., for $R_1 < r < R_2$ and $L_1 < z < L_2$. Using (18) with (20) in (19), we obtain the vector potential in the region that the coil occupies as

$$A_{n\text{-turn}} = \frac{\mu_0 J}{\pi} \int_0^\infty \frac{2}{\alpha} \sin \frac{\alpha L}{2} \cos \alpha(z - L_0) S(\alpha, r)\, d\alpha \tag{21}$$

where $L_0 = (L_1 + L_2)/2$ and $L = L_2 - L_1$ is the length of the coil, and

$$S(\alpha, r) = \int_{R_1}^{R_2} r_0 Q(\alpha, r, r_0)\, dr_0. \tag{22}$$

Induced voltage for a single-turn coil of radius r_0 is given by the line integral over the coil:

$$V = i\varpi \oint_{coil} \vec{A} \cdot d\vec{r} = i\varpi 2\pi r_0 A(r_0, z_0).$$

For the n-turn coil, use superposition:

$$V = \frac{i\varpi 2\pi n}{(L_2 - L_1)(R_2 - R_1)} \int_{L_1}^{L_2} \int_{R_1}^{R_2} r A_{n\text{-turn}}(r, z)\, dr\, dz. \tag{23}$$

Here, it is assumed that the turns are uniformly distributed over the cross section. The coil impedance $Z = V/I$ is found by substituting (21) in (23) as

$$Z = Z_0 + \frac{8 i\varpi \mu_0 n^2}{L^2 (R_2 - R_1)^2} \int_0^\infty \frac{1}{\alpha^2} \sin^2 \frac{\alpha L}{2} \phi(\alpha) P^2(\alpha)\, d\alpha \tag{24}$$

where

$$P(\alpha) = \int_{R_1}^{R_2} x K_1(\alpha x)\, dx \tag{25}$$

and Z_0, the impedance of the n-turn coil in free-space, is given by

$$Z_0 = \frac{16 i\varpi \mu_0 n^2}{L^2 (R_2 - R_1)^2} \int_0^\infty \frac{1}{\alpha^2} \sin^2 \frac{\alpha L}{2} \cdot \left\{ \int_{R_1}^{R_2} \int_y^{R_2} x y K_1(\alpha x) I_1(\alpha y)\, dx\, dy \right\} d\alpha. \tag{26}$$

We are interested in the impedance difference between a layered cylinder and a homogeneous cylinder of conductivity σ_1. The reason for this, as explained in [7], is to subtract some of the modeling errors. Since for a homogeneous cylinder of radius r_{N-2} and conductivity σ_1

$$\phi_{\text{hom}}(\alpha) = \frac{\alpha I_0(\alpha r_{N-2}) I_1(\alpha_1 r_{N-2}) - \alpha_1 I_0(\alpha_1 r_{N-2}) I_1(\alpha r_{N-2})}{\alpha K_0(\alpha r_{N-2}) I_1(\alpha_1 r_{N-2}) + \alpha_1 K_1(\alpha r_{N-2}) I_0(\alpha_1 r_{N-2})} \tag{27}$$

the impedance difference $\Delta Z = Z_{\text{homog}} - Z_{\text{layered}}$ is given by

$$\Delta Z = \frac{8 i \varpi \mu_0 n^2}{L^2 (R_2 - R_1)^2} \int_0^\infty \frac{1}{\alpha^2} \sin^2 \frac{\alpha L}{2} \cdot [\phi_{\text{hom}}(\alpha) - \phi(\alpha)] P^2(\alpha)\, d\alpha. \tag{28}$$

Finally, we give an explicit formula for computing the vector potential in layer m in the case of an n-turn coil

$$A^{(m)} = \frac{2\mu_0 n I}{\pi L (R_2 - R_1)} \int_0^\infty \{[\phi(\alpha) q_{m11} + q_{m12}] K_1(\alpha_m r) + [\phi(\alpha) q_{m21} + q_{m22}] I_1(\alpha_m r)\} \cdot \frac{P(\alpha)}{\alpha} \sin \frac{\alpha L}{2} \cos \alpha (z - z_0)\, d\alpha \tag{29}$$

where

$$\begin{bmatrix} q_{m11} & q_{m12} \\ q_{m21} & q_{m22} \end{bmatrix} = H_m^{-1} H_{m+1}^{-1} \cdots H_{N-2}^{-1} \tag{30}$$

and the inverse of H_m is given by

$$H_m^{-1} = r_m \begin{bmatrix} a_{m22} & -a_{m12} \\ -a_{m21} & a_{m11} \end{bmatrix}. \tag{31}$$

III. Numerical Method for a Continuous Conductivity Profile

The solution given in the previous section can be used to compute the impedance change for any continuously changing radial conductivity profile. First, the profile is divided into $N-2$ discrete regions. By taking the conductivity in region k ($r_k < r < r_{k+1}$) equal to the constant

$$\sigma\left(\frac{r_k + r_{k+1}}{2}\right) = \sigma_k$$

we obtain an approximate piecewise constant profile to which the solution of the previous section can be applied. As the number of layers, $N-2$, is increased, the solution should converge to the exact solution for the continuous profile.

As a test case for the method, we consider the following conductivity change with the radial distance inside the workpiece

$$\sigma = a + b \tanh\left(\frac{r-c}{\lambda}\right) \tag{32}$$

where

$$a = \frac{\sigma_c \tanh\left(\frac{R-c}{\lambda}\right) + \sigma_s \tanh\left(\frac{c}{\lambda}\right)}{\tanh\left(\frac{c}{\lambda}\right) + \tanh\left(\frac{R-c}{\lambda}\right)} \tag{33}$$

TABLE I
Fixed Parameters for the Sample Computations

Inner radius of coil	r_1=12 mm
Outer radius of coil	r_2=15 mm
Length of coil	L=30 mm
Number of turns	n=500
Radius of the cylinder	R=10 mm
Surface conductivity	σ_s=10^7 S/m
Conductivity at the center	σ_c=2.10^7 S/m
Inflection point	c=8 mm

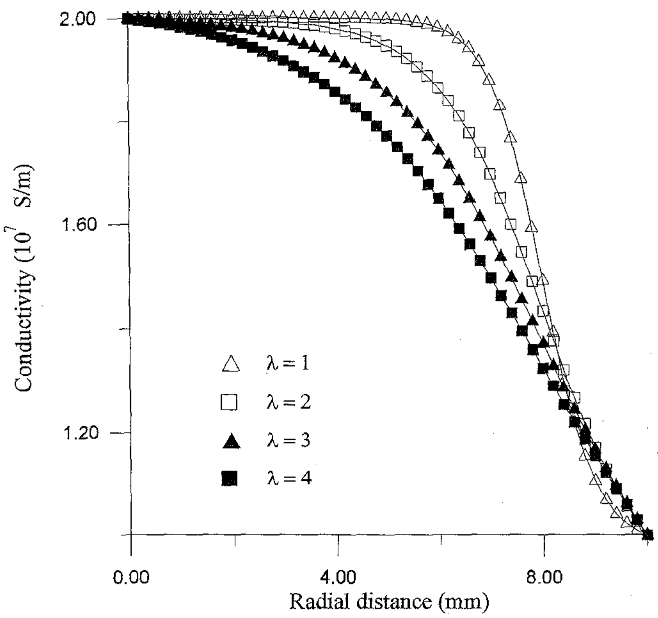

Fig. 3. Sample conductivity profiles described by (32).

and

$$b = -\frac{\sigma_c - \sigma_s}{\tanh\left(\frac{c}{\lambda}\right) + \tanh\left(\frac{R-c}{\lambda}\right)} \tag{34}$$

where σ_c and σ_s are the conductivities at the center and at the surface of the cylinder, respectively. Equation (32) shows a continuous change in conductivity from σ_c to σ_s, the steepness of the change being controlled by the parameter λ, which has the dimension of length. The inflection point in the profile is at $r = c$, for $\lambda = 0$, (32) becomes a discontinuous change at $r = c$. R is the radius of the cylindrical workpiece.

In order to give some numerical results, we take the parameters of the problem as shown in Table I. The resulting conductivity profiles for various values of λ are shown in Fig. 3. In this case, the homogeneous or unprocessed cylinder has uniform conductivity σ_c everywhere.

To examine the convergence of the method, when the number of layers is increased, we consider the $\lambda = 2$ mm case in Table I and divide the profile into 10, 20, 30, $\cdots$ layers. The computed impedance difference at a frequency of 1000 Hz converges as shown in Table II. We found that taking about

TABLE II
CONVERGENCE OF THE METHOD AT 1 kHz

Number of Layers	Resistance (ohms)	Reactance (ohms)
10	-0.0937	-1.4438
20	-0.0853	-1.4588
30	-0.0838	-1.4615
40	-0.0832	-1.4625
50	-0.0830	-1.4629

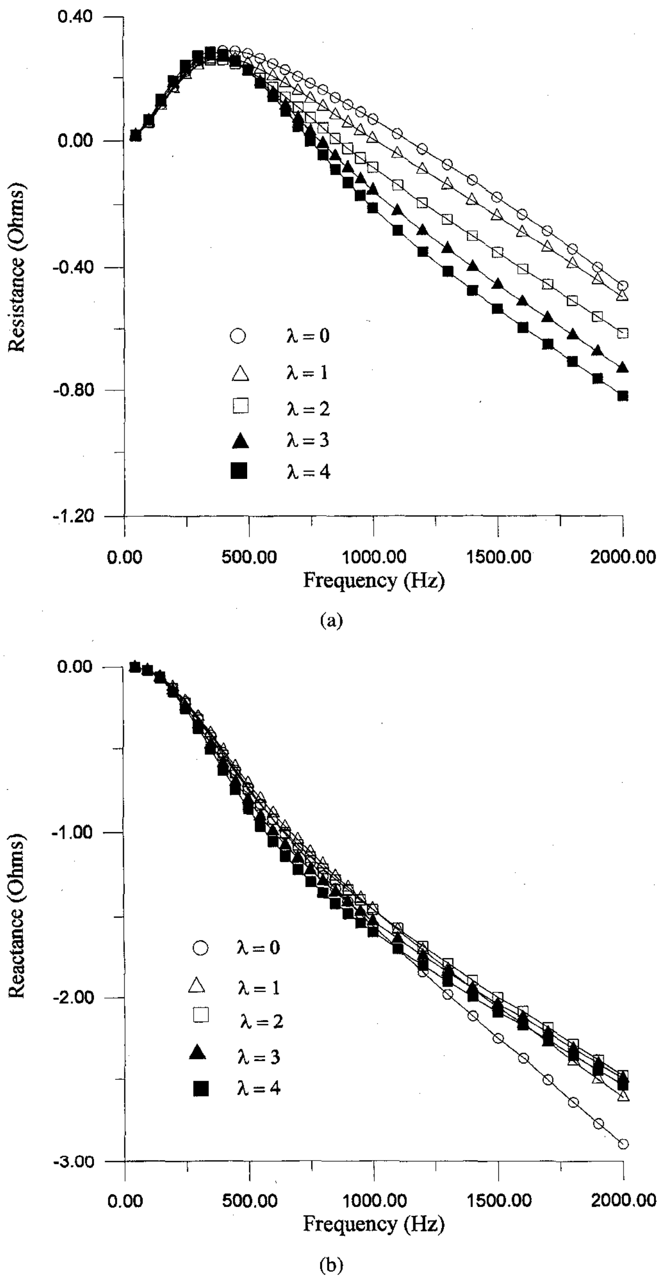

Fig. 4. (a) Real part of the impedance as a function of frequency for the profiles of Fig. 3. (b) Imaginary part of the impedance as a function of frequency for the profiles of Fig. 3.

40–50 discrete layers is enough for four-place accuracy in the computed impedance difference.

Real and imaginary parts of the impedance differences for the profiles in Fig. 3 are shown in Fig. 4(a) and (b). It is clear that, on the average, the real part is more sensitive to the shape of the profile than the imaginary part. Also, there is a characteristic relative maxima and a zero crossing in the real part, as was the case for the planar surface layers [6], [7]. As the steepness of the profile is changed, the relative maxima does not change much, but the zero crossing frequency strongly depends on the steepness of the profile.

IV. CONCLUSION

We described a method to compute the impedance of a multiturn coil surrounding a cylindrical workpiece that has an arbitrary radial conductivity variation. The method was based on an analytical solution for the problem with the same geometry, but with a piecewise constant-radial conductivity. We presented sample computations using the proposed method for several conductivity profiles. This method has obvious advantages over the purely numerical methods, namely, we do not have to do any discretization except the 1-D conductivity profile. Also, instead of solving large systems of linear algebraic equations, we need to evaluate well-known special functions and compute some integrals numerically.

REFERENCES

[1] D. H. S. Cheng, "The reflected impedance of a circular coil in the proximity of a semi-infinite medium," *IEEE Trans. Instrum. Meas.*, vol. IM-14, pp. 107–116, 1965.
[2] C. V. Dodd and W. E. Deeds, "Analytical solutions to eddy-current probe-coil problems," *J. Appl. Phys.*, vol. 39, pp. 2829–2838, 1968.
[3] C. C. Cheng, C. V. Dodd, and W. E. Deeds, "General analysis of probe coils near stratified conductors," *Int. J. Nondestructive Test.*, vol. 3, pp. 109–130, 1971.
[4] E. Uzal, J. C. Moulder, S. Mitra, and J. H. Rose, "Impedance of coils over layered metals with continuously variable conductivity and permeability: Theory and experiment," *J. Appl. Phys.*, vol. 74, no. 3, Aug. 1993.
[5] S. J. Norton, A. H. Kahn, and M. L. Mester, "Reconstructing electrical conductivity profiles from variable-frequency eddy current measurements," *Res. Nondestructive Eval.*, vol. 1, pp. 167–179, 1989.
[6] J. C. Moulder, E. Uzal, and J. H. Rose, "Thickness and conductivity of metallic layers from eddy current measurements," *Rev. Sci. Instrum.*, vol. 63, no. 6, June 1992.
[7] E. Uzal, J. C. Moulder, and J. H. Rose, "Experimental determination of the near-surface conductivity profiles from electromagnetic induction (eddy–current) measurements," *Inverse Probl.*, vol. 9, 1994.

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