STABILITY AND SPATIO-TEMPORAL BEHAVIOUR OF COUPLED MAPS WITH DELAYS

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- CML's introduced by Kaneko in 80's
- Coupled maps
  - networks
  - graph theory
  - time delay
- Synchronization
  - a universal concept in nonlinear sciences
  - an important phenomena in coupled systems
- Numerical $>>$ Analytical
- Our aim:
  - to obtain necessary and sufficient condition for stability of coupled maps
  - to get some idea about more complex solutions by bifurcation analysis

- STABILITY OF THE FIXED POINT AND STABILITY REGION IN PARAMETER SPACE
- DEPENDENCE OF SR ON DELAY
- DEPENDENCE OF SR ON CONNECTION TOPOLOGY
- STABILIZATION OF MAPS BY COUPLING
- BIFURCATIONS

STABILITY OF THE FIXED POINT
AND
STABILITY REGION IN PARAMETER SPACE (b, ε)

Dynamics of coupled maps:

$$x_i(t+1) = f(x_i(t)) + \varepsilon \frac{1}{n_i} \sum_{\substack{j \\ j \sim i}} (f(x_j(t-\tau)) - f(x_i(t))) \tag{1}$$

Linearized equation for eigenvector of Laplacian u_k:

$$u(t+1) = (1-\varepsilon) b u(t) + \varepsilon(1-\lambda_k) b u(t-\tau) \tag{2}$$

where $b = f'(x^*)$ and λ_k is corresponding eigenvalue of Laplacian.

Characteristic polynomial for eigenvector u_k:

$$p_k(s) = s^{\tau+1} - (1-\varepsilon) b s^{\tau} - \varepsilon(1-\lambda_k) b \tag{3}$$

Remark: The results are from F. M. Atay, J. Jost, A. Wende, Phys. Rev. Let.,2004

Theorem 1 *Let b and ε are arbitrary real numbers. Then the fixed point X^* of (1) is asymptotically stable for all $\tau \in \mathbb{Z}^+$ if $|b| < 1$ and $|b||1 - 2\varepsilon| < 1$.*

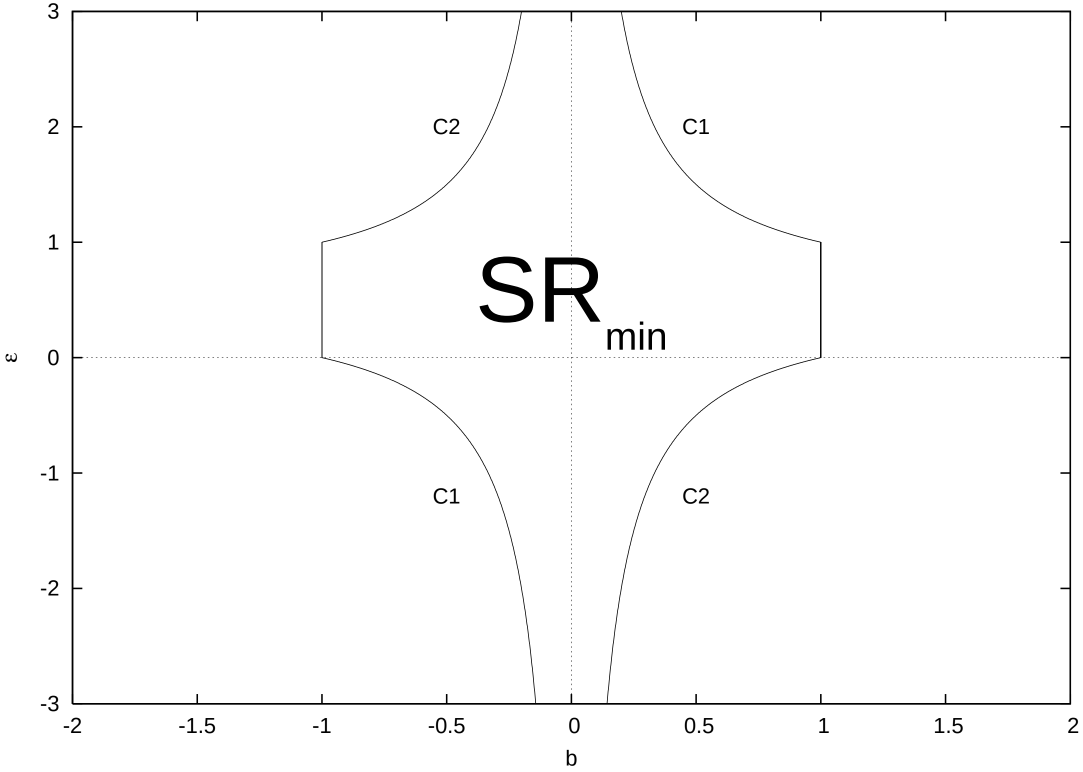

Figure 1: minimal stability region

Theorem 2 *X^* is asymptotically stable if and only if for smallest and largest eigenvalues of the Laplacian ,namely , for $\lambda_k = 0$ and $\lambda_k = \lambda_{max}$, one of the following hold:*

(i) τ is odd and

$$|(1-\varepsilon)b| - 1 < -\varepsilon(1-\lambda_k)b < \sqrt{(1-\varepsilon)^2 b^2 + 1 - 2|(1-\varepsilon)b|\cos\Phi} \tag{4}$$

or

$$\textit{(ii) } \tau \textit{ is even, } |b||1-\varepsilon\lambda_k| < 1 \tag{5}$$

$$\textit{and } |\varepsilon b| < \sqrt{(1-\varepsilon)^2 b^2 + 1 - 2|(1-\varepsilon)b|\cos\Phi} \tag{6}$$

where Φ is the unique solution of $\sin((\tau+1)\Phi)/\sin(\tau\Phi) = |1-\varepsilon||b|$ in the interval $(0, \pi/(\tau+1))$.

DEPENDENCE OF SR ON DELAY

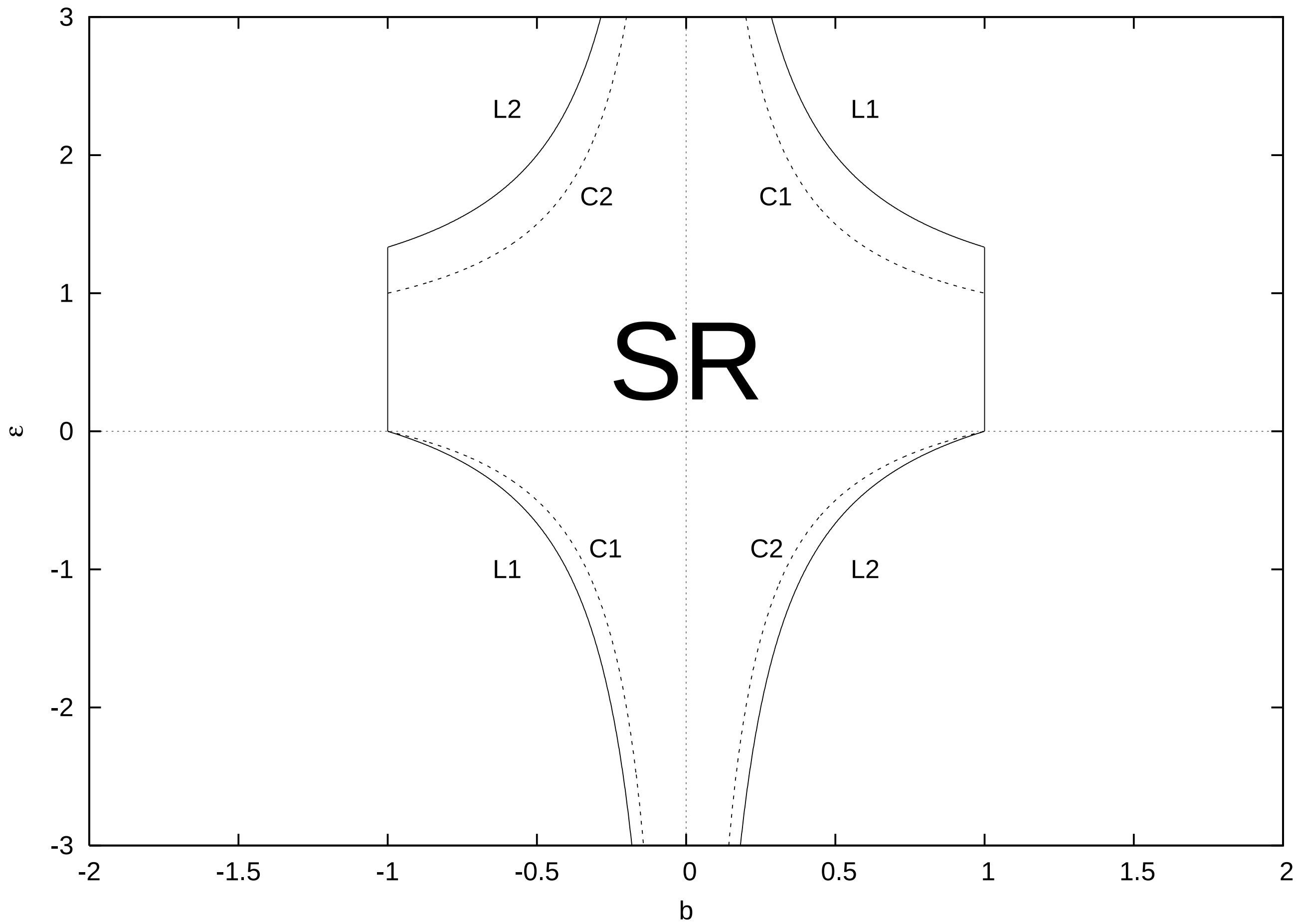

Figure 2: stability region for $\tau = 0$ ($\lambda_{max} = 1.5$)

(The dotted lines represent bounds of minimal stability region)

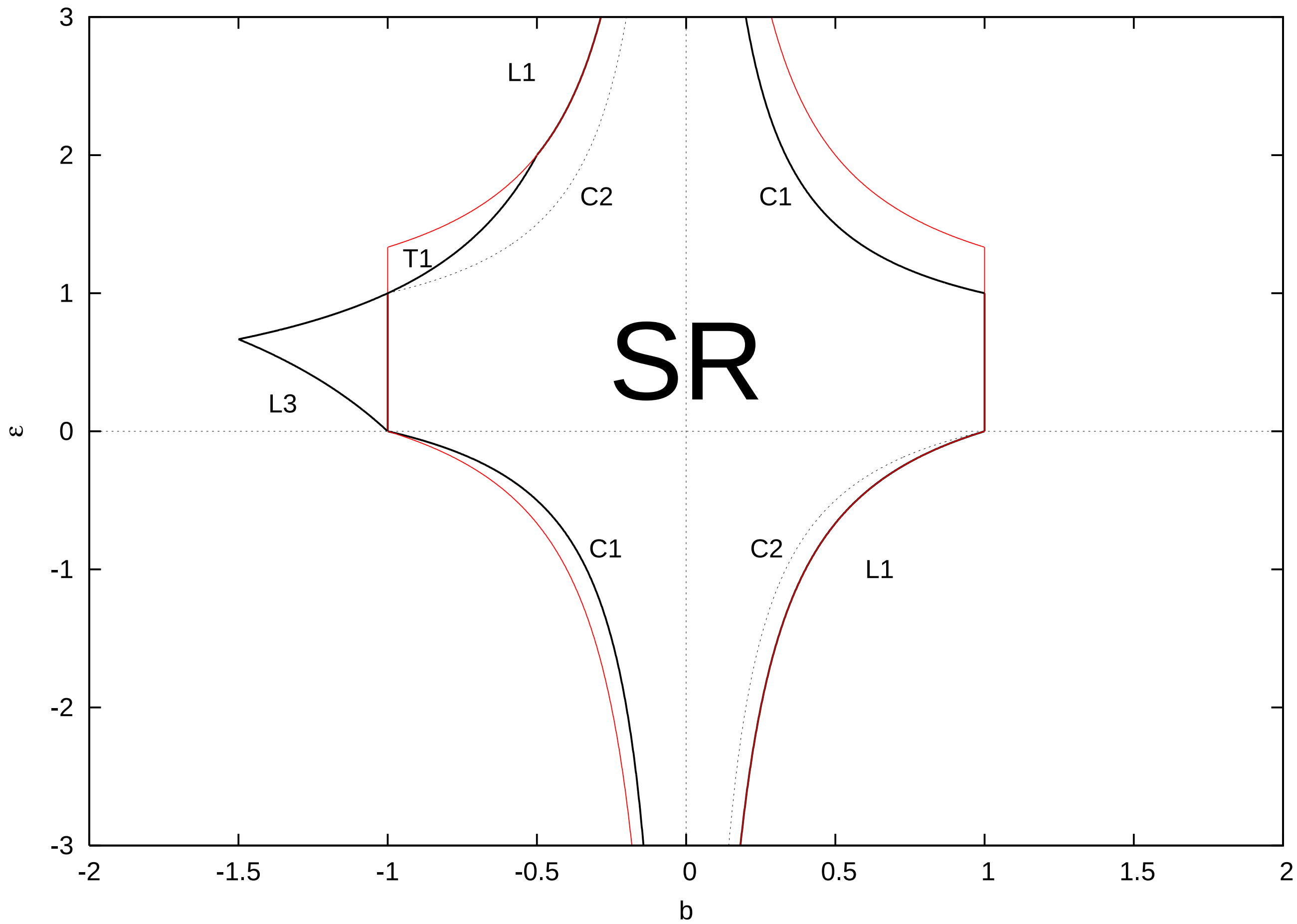

Figure 3: stability region for $\tau = 1$ ($\lambda_{max} = 1.5$)

(The dotted and red lines represent bounds of minimal stability region and the stability region for $\tau = 0$, respectively.)

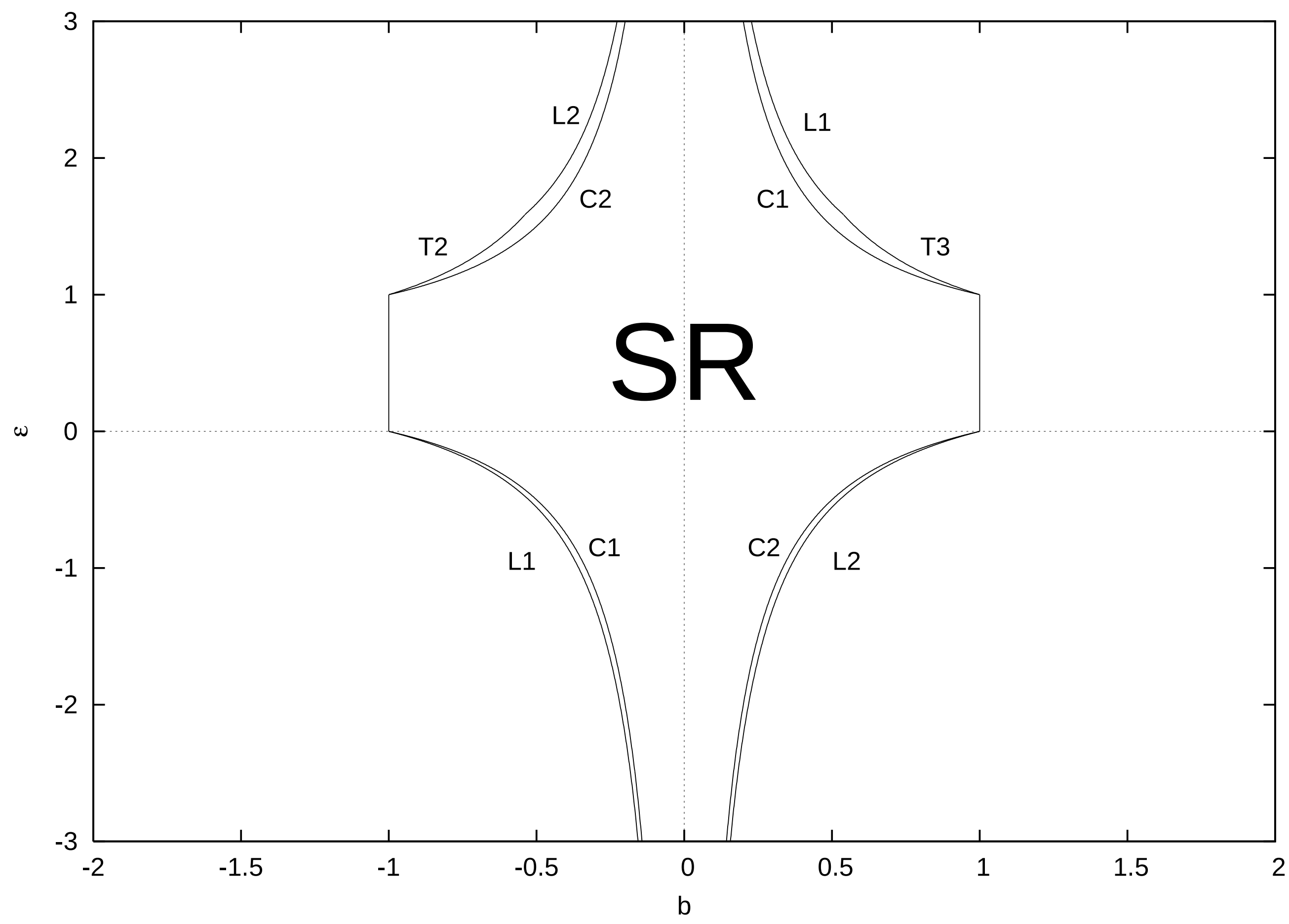

Figure 4: stability region for $\tau = 2$ ($\lambda_{max} = 1.8$)

(The dotted lines represent bounds of minimal stability region)

Corollary 3 *Let*

$$SR_{\tau',\lambda'} = \{(b,\varepsilon)\Big| X^* \text{ is asymptotically stable for } \tau = \tau' \text{ and } \lambda_{max} = \lambda'\} \tag{7}$$

and

$$SR_{min} = \{(b,\varepsilon)\Big| |b| < 1 \text{ and } |b||1-2\varepsilon| < 1\}. \tag{8}$$

Then $SR_{\tau,\lambda} \to SR_{min}$ *as* $\tau \to \infty$ *for each* $\lambda \in [1,2]$

DEPENDENCE OF SR ON CONNECTION TOPOLOGY

Corollary 4 *Let* Γ_A *and* Γ_B *be two graphs with largest eigenvalues* λ^A_{max} *and* λ^B_{max}*, respectively, such that* $\lambda^B_{max} < \lambda^A_{max}$*. If the fixed point of (1) is asymptotically stable in* Γ_A*, then it is also asymptotically stable in* Γ_B*.*

Remark:A similar result holds for continuous time delay systems.(Atay,J. Diff. Eq.)

$$SR_{\tau',\lambda'} = \{(b,\varepsilon) \Big| X^* \text{ is asymptotically stable for } \tau = \tau' \text{and } \lambda_{max} = \lambda'\} \tag{9}$$

$$SR_{min} = \{(b,\varepsilon) \Big| |b| < 1 \text{ and } |b||1-2\varepsilon| < 1\} \tag{10}$$

Corollary 5 *Let $SR_{\tau,\lambda}$ and SR_{min} the stability regions of (1) defined by Eq. (9) and Eq. (10),respectively. Then $SR_{\tau,2} = SR_{min}$ for all $\tau \in \mathbb{Z}^+$.*

STABILIZATION OF MAPS BY COUPLING

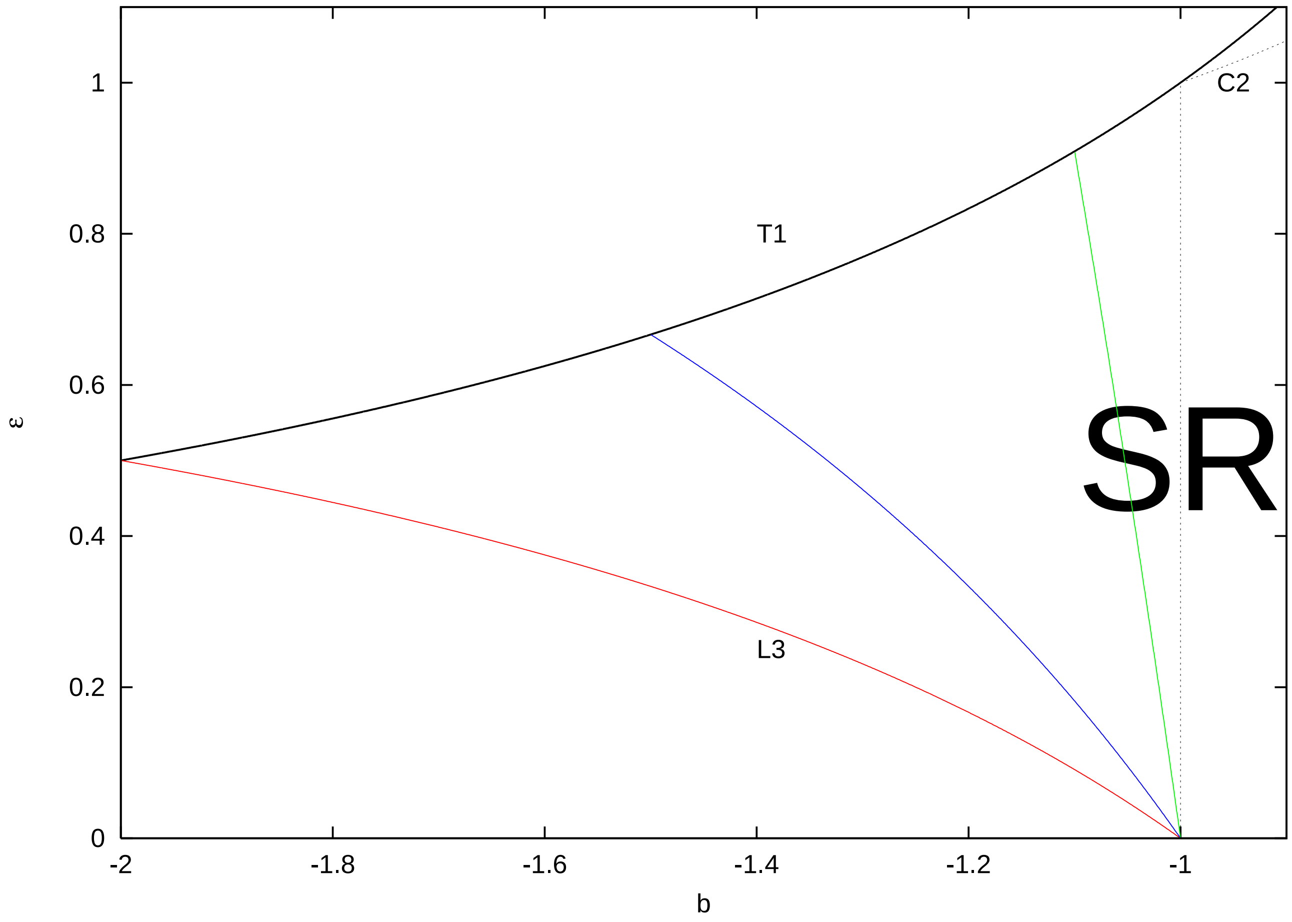

Figure 5: A part of the stability region for $\tau = 1$ ($\lambda_{max} = 1.5$)

(The dotted lines represent bounds of minimal stability region and the red, blue and green lines represent bounds for $\lambda_{max} = 1, 1.5, 1.9$ respectively.)

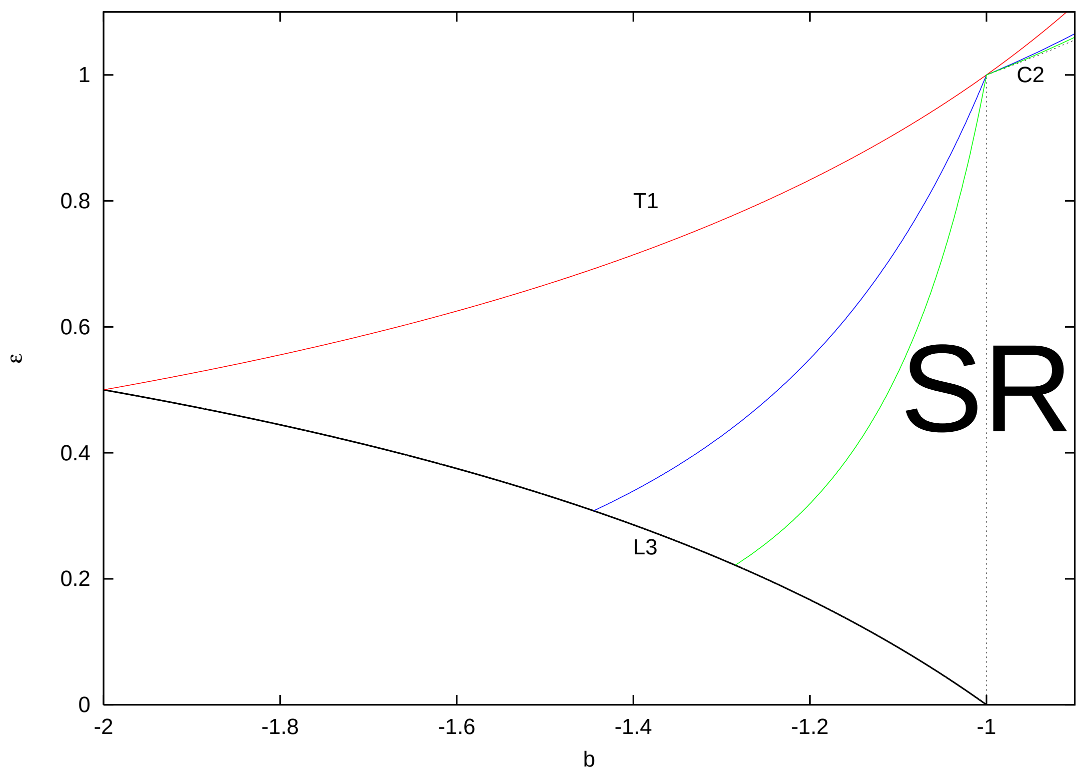

Figure 6: A part of the stability region for $\lambda_{max} = 1$ and for odd delays.

(The dotted lines represent bounds of minimal stability region and the red, blue and green lines represent bounds for $\tau = 1, 3, 5$ respectively.)

Corollary 6 *Unstable identical maps* $x(t+1) = f(x(t))$ *can be stabilized by coupling as in (1) with* $\tau = 1$ *if and only if* $\lambda_{max} - 3 < f'(x^*) < -1$, *where* λ_{max} *is the maximum eigenvalue of Laplacian. Moreover, for stabilizing coupling strength should satisfy following condition:*

$$\frac{b+1}{b(2-\lambda_{max})} < \varepsilon < -\frac{1}{b}$$

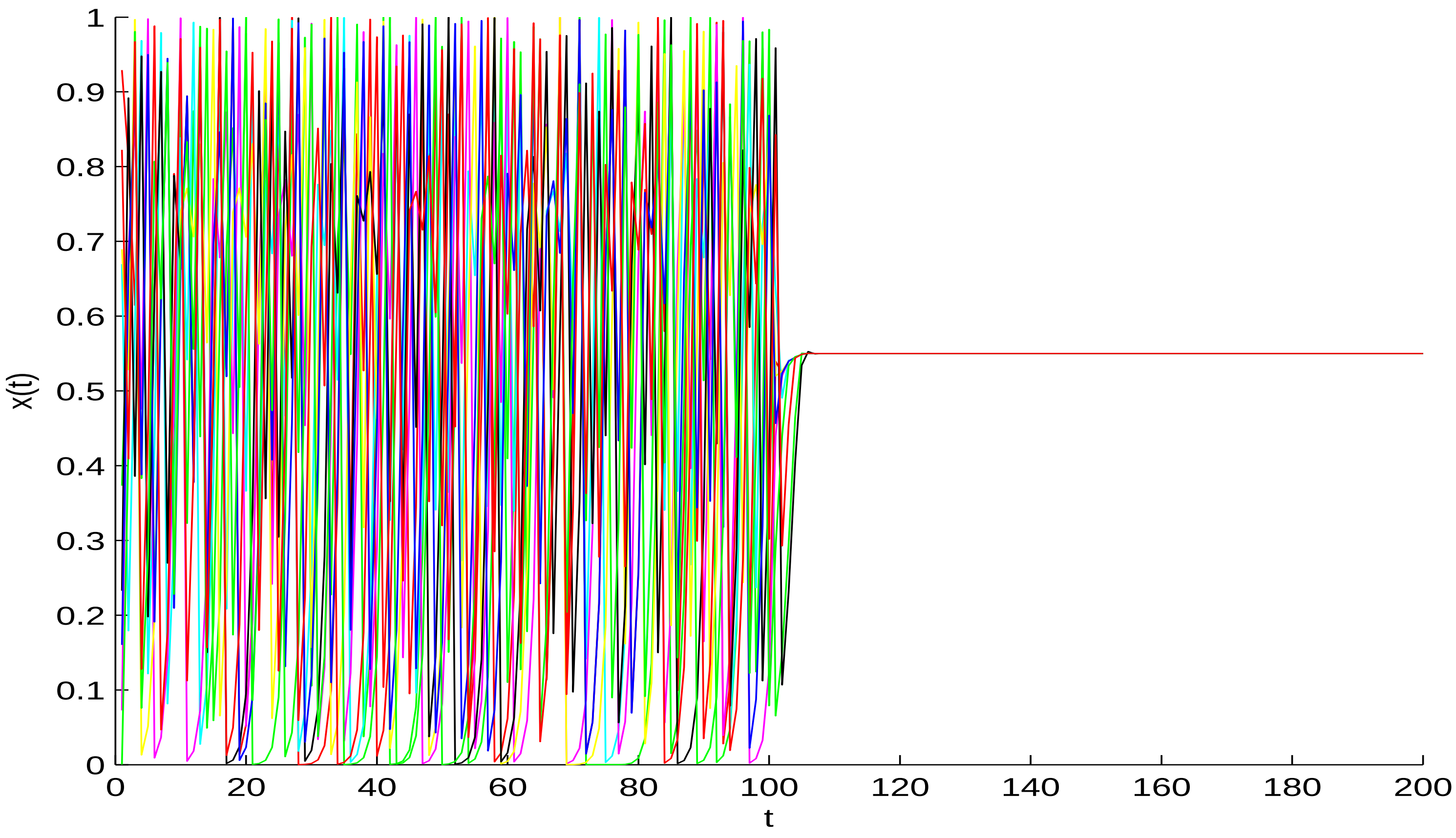

Figure 7: Chaotic logistic maps (uncoupled before 100, coupled with $\lambda_{max} = 1$ and $\tau = 1$ after 100)

BIFURCATIONS

$$p_k(s) = s^{\tau+1} - (1-\varepsilon)bs^{\tau} - \varepsilon(1-\lambda_k)b \tag{11}$$

$$p(s) = \prod_k p_k(s) \tag{12}$$

Flip: $s = -1$

Fold: $s = 1$

Neimark-Sacker: $s = a + ib,\ b \neq 0$

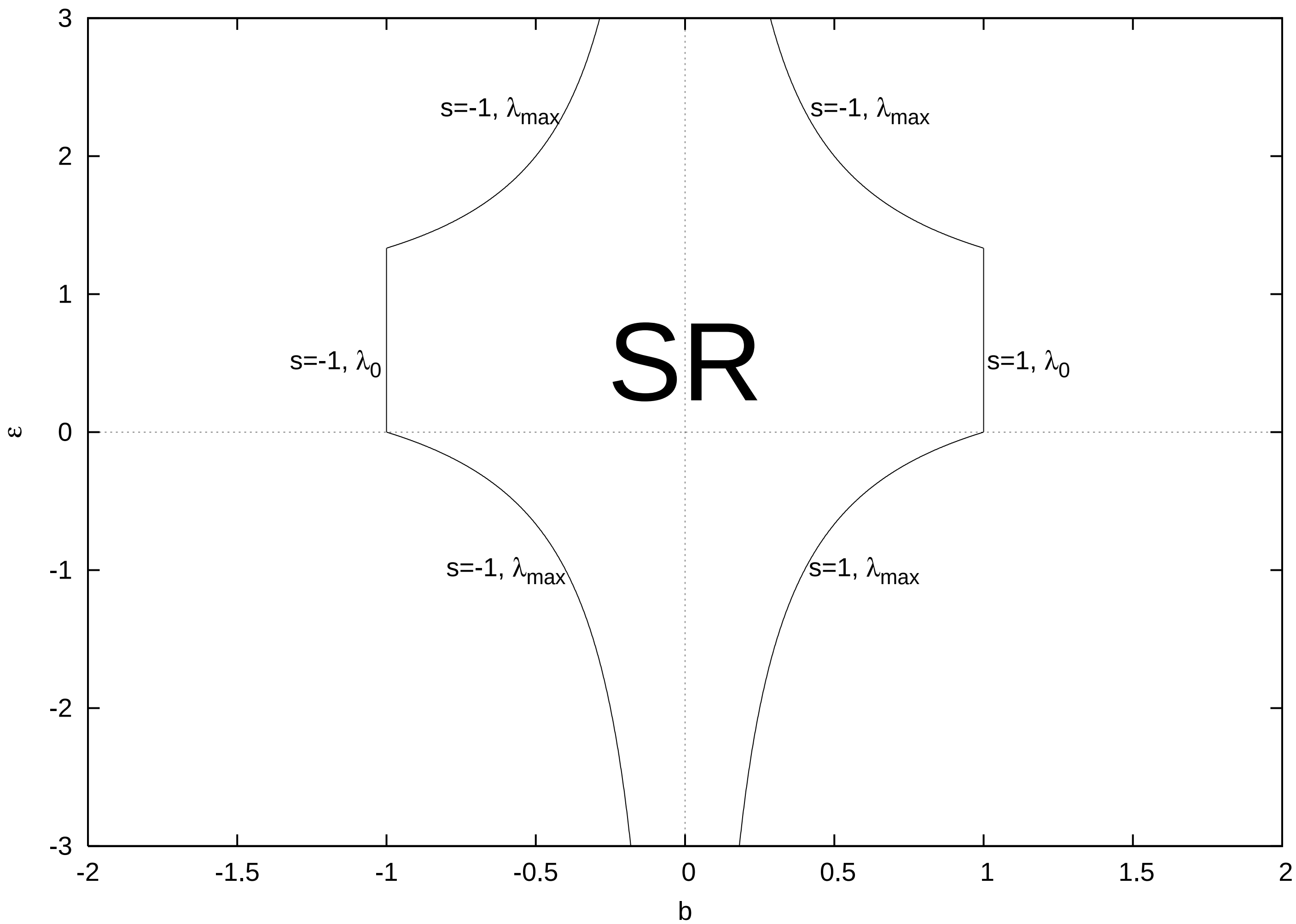

Figure 8: Bifurcation curves for $\tau = 0$ ($\lambda_{max} = 1.5$)

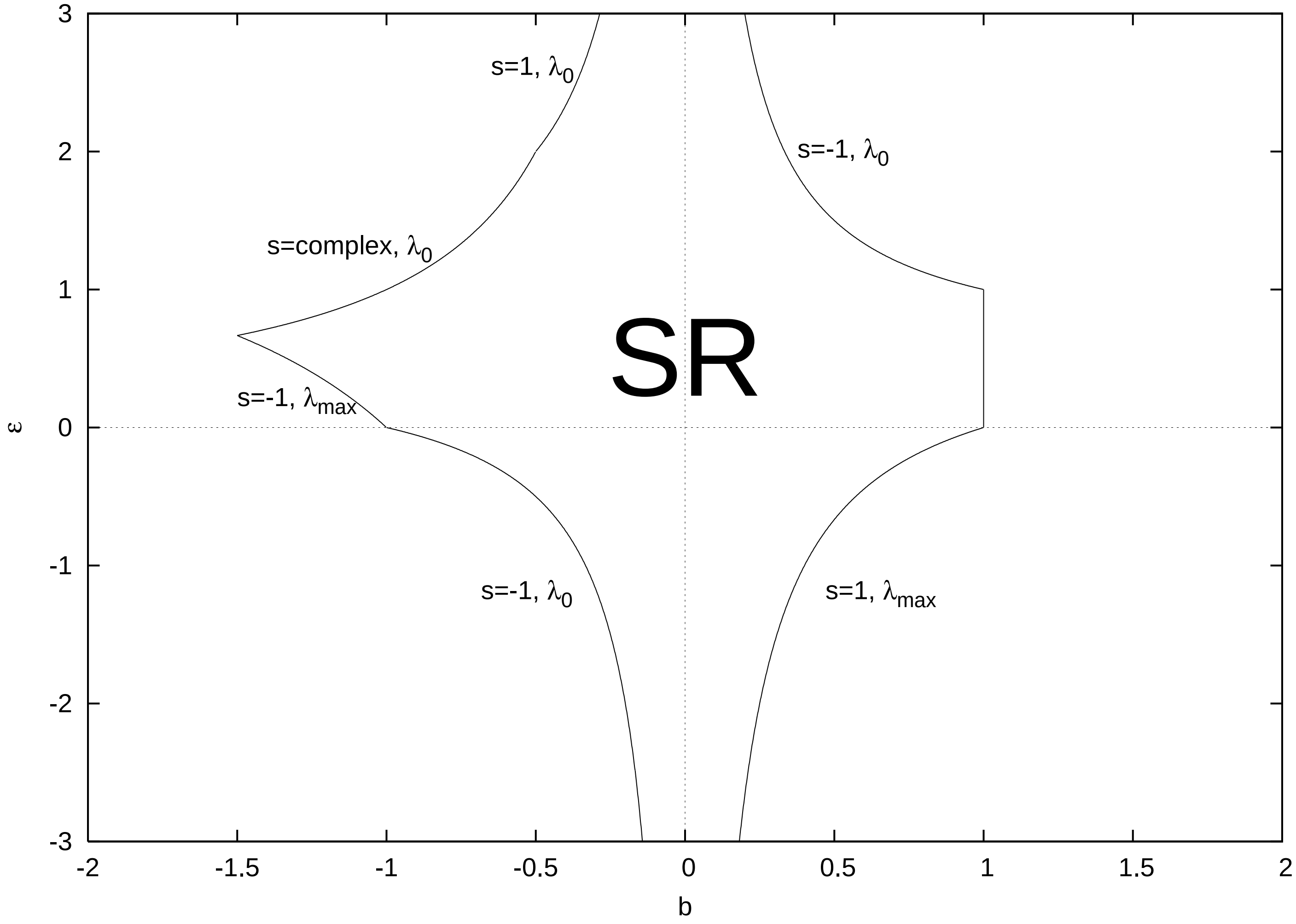

Figure 9: Bifurcation curves for $\tau = 1$ ($\lambda_{max} = 1.5$)

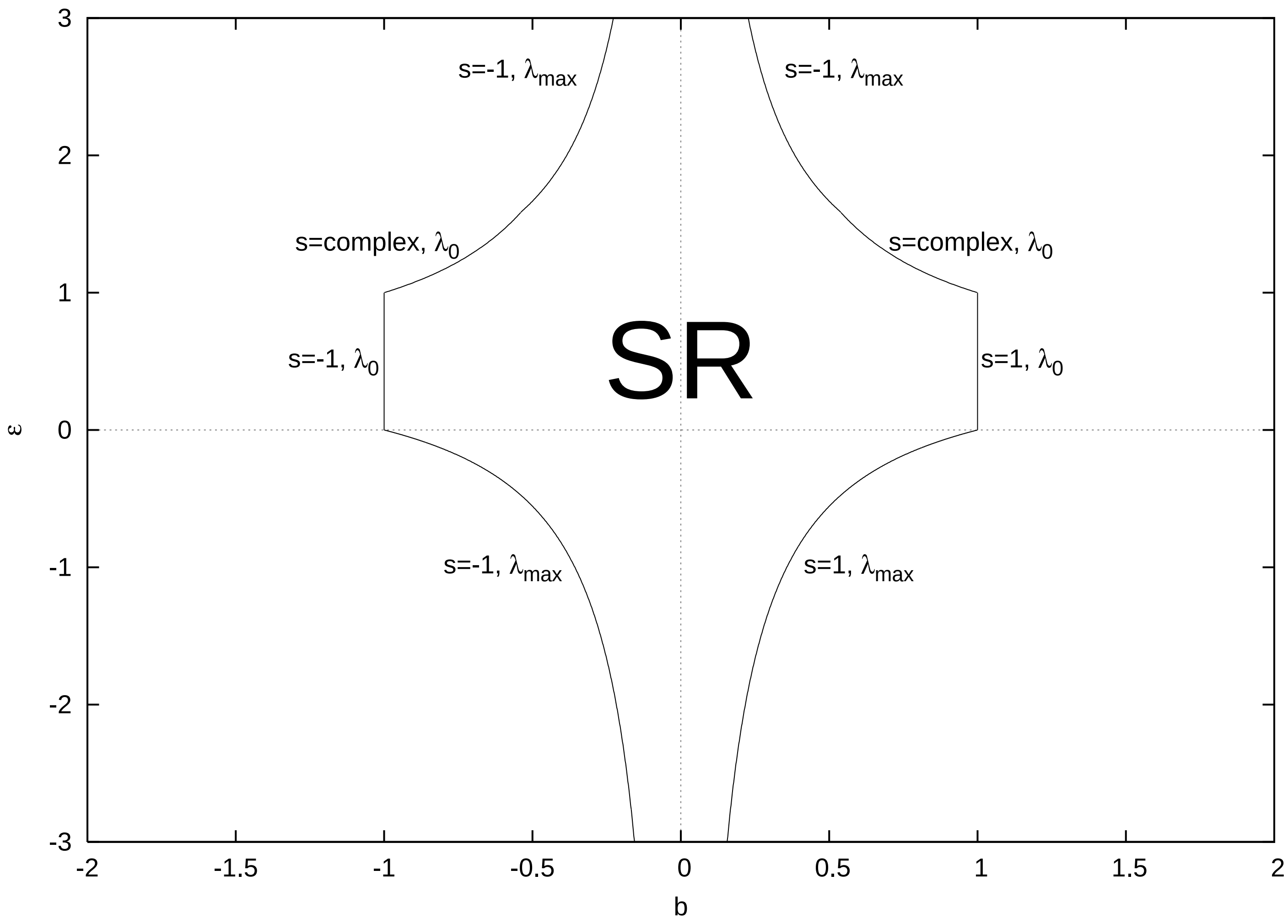

Figure 10: Bifurcation curves for $\tau = 2$ ($\lambda_{max} = 1.8$)

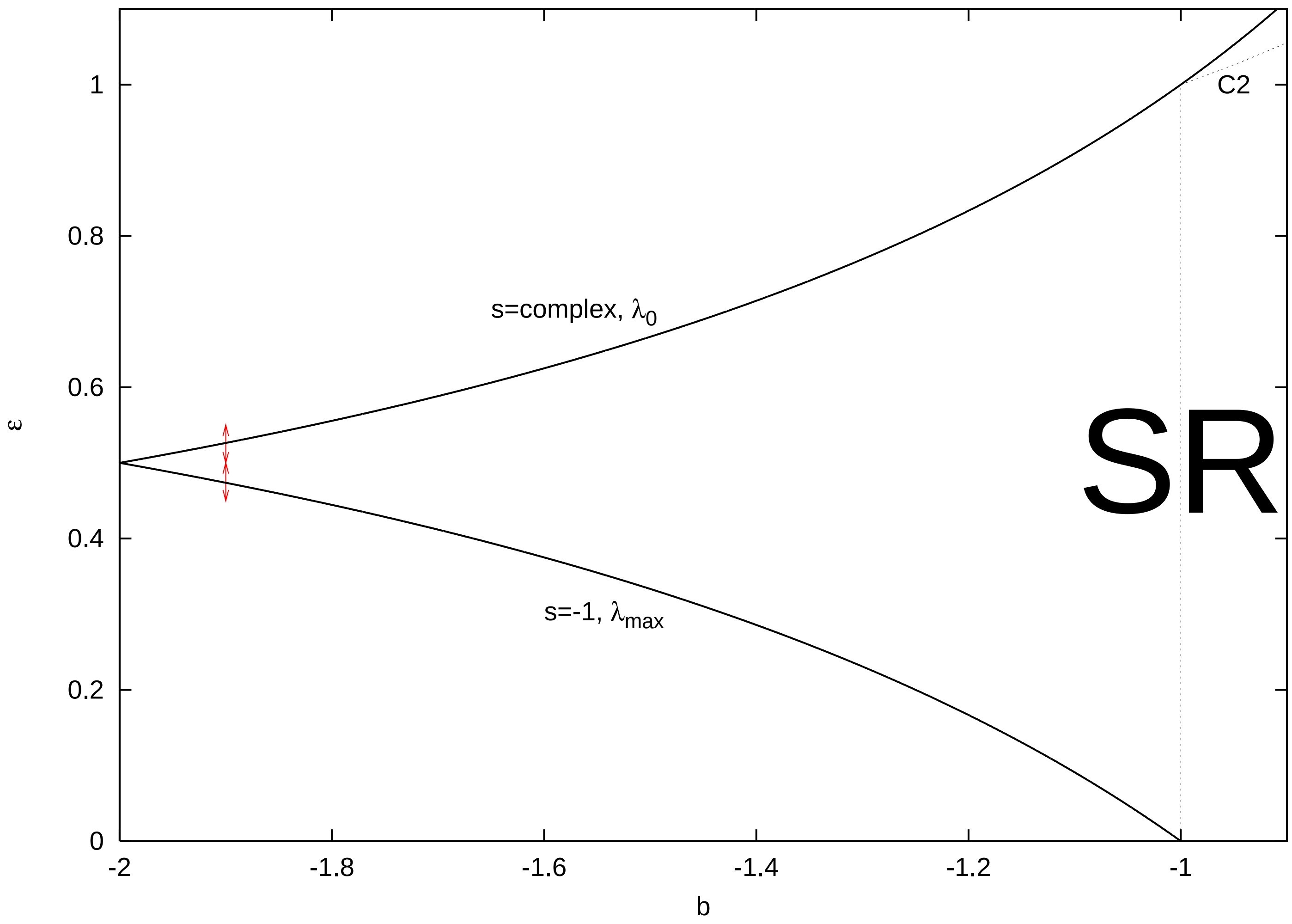

Figure 11: An example of bifurcations for $\tau = 1$ ($\lambda_{max} = 1$)

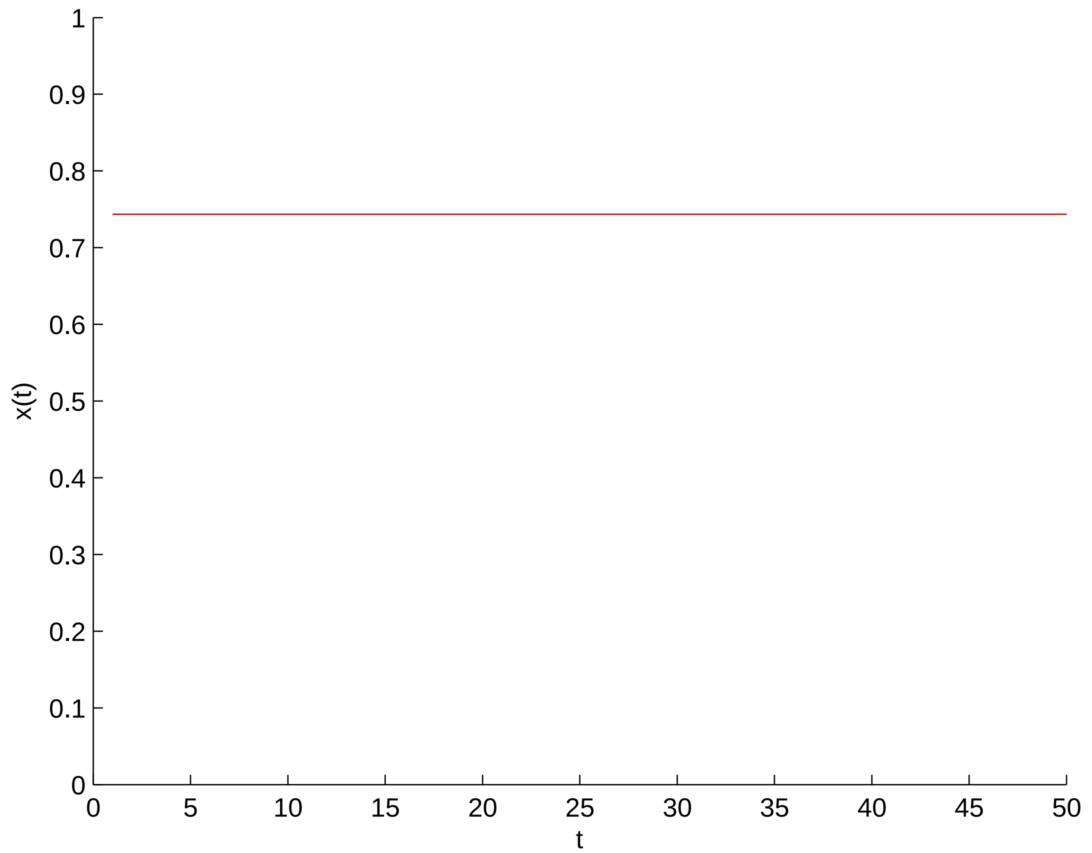

Figure 12: Activation of 10 cells $a = 3.9$, $\varepsilon = 0.5$

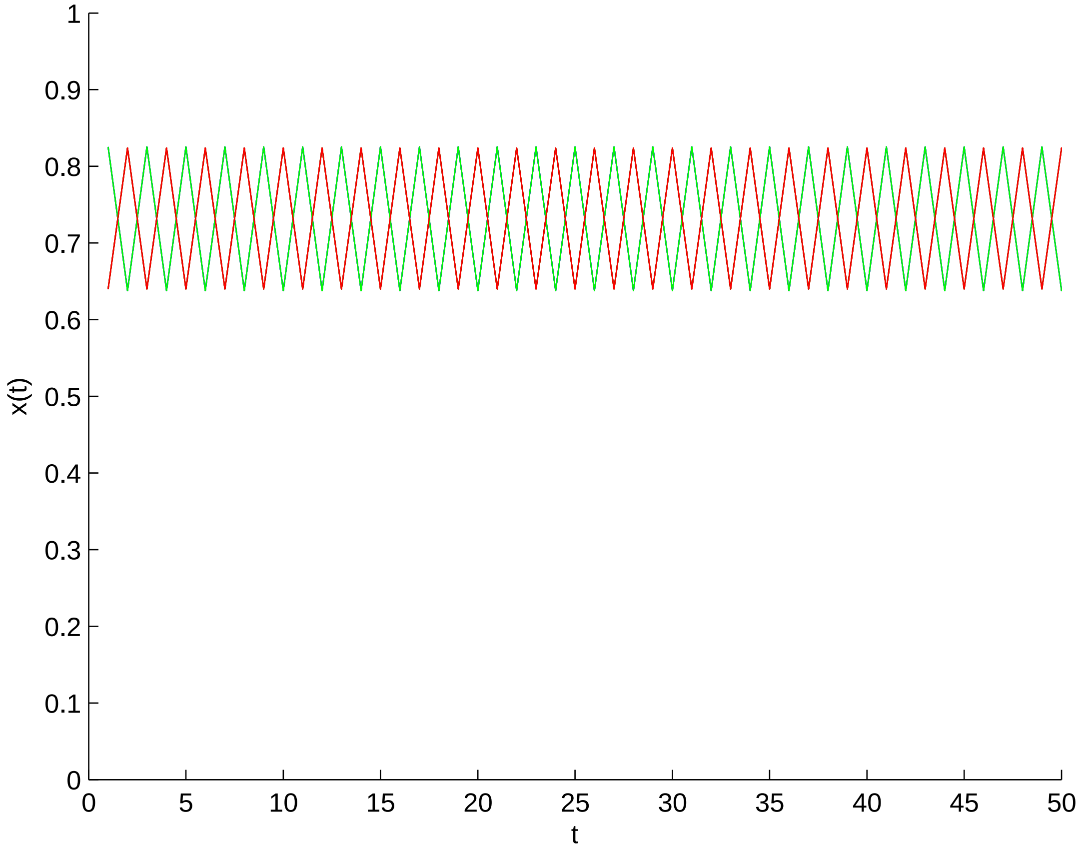

Figure 13: Activation of 10 cells $a = 3.9$, $\varepsilon = 0.45$

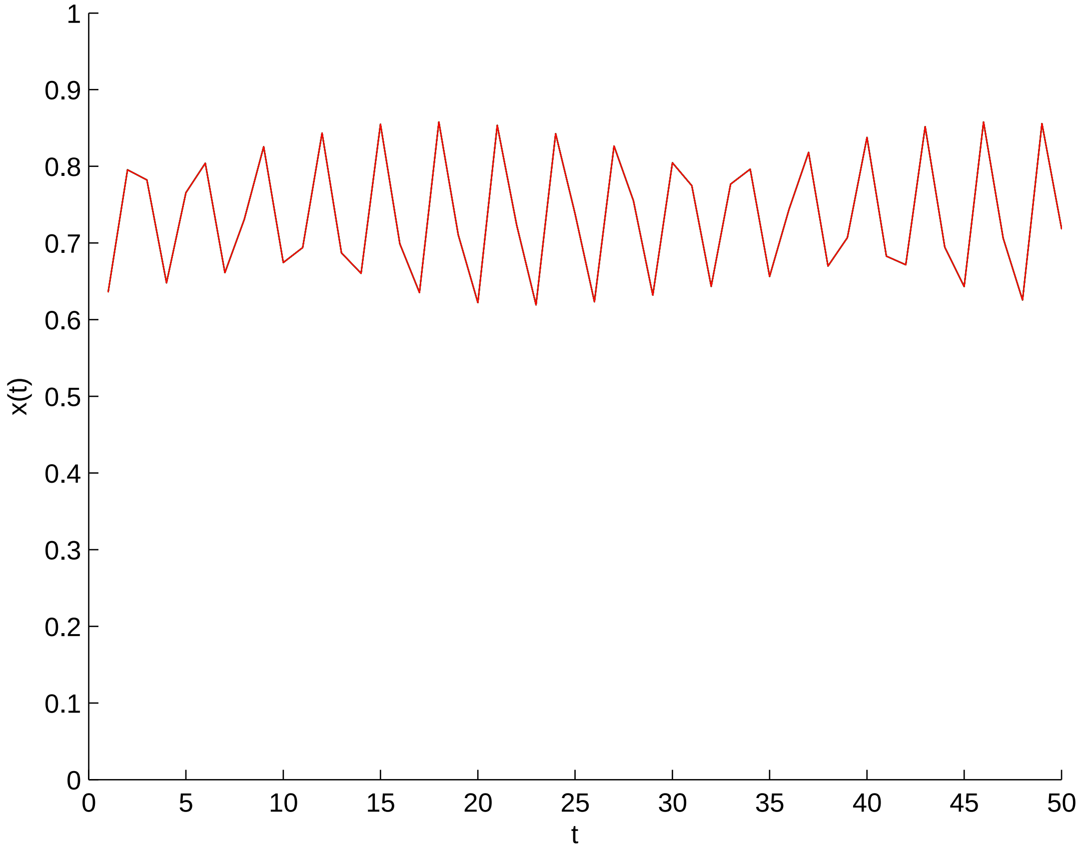

Figure 14: Activation of 10 cells $a = 3.9$, $\varepsilon = 0.55$

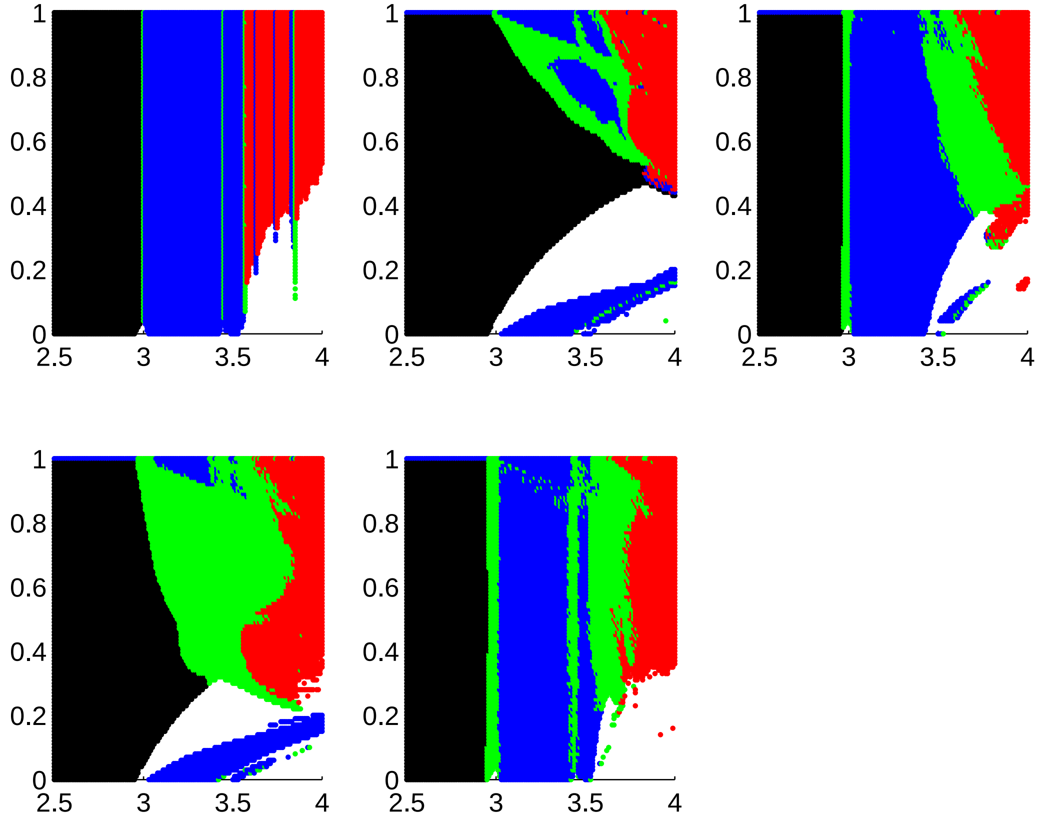

Figure 15: Coupled logistic maps for $\lambda_{max} = 1$ and $\tau = 0, 1, 2, 3, 4$ respectively

CONCLUSION

- Complete stability analysis of fixed points of coupled map with and without delays and bifurcation analysis

- Study of delay is important, since it has significant effects on dynamics

- Stabilization of individual maps by coupling and chaos suppression

- The effect of connection topology on the stability of the network