OPERATOR SPLITTING METHODS FOR COMPUTATION OF EIGENVALUES OF REGULAR STURM-LIOUVILLE PROBLEMS

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13/06/2016

Sturm J.C.F

Liouville J.

Outline

Thales
BC 624-546

Introduction

We discuss the computation of higher eigenvalues of regular Sturm-Liouville problem (SLP) in canonical Liouville normal form

$$-y''(t) + q(t)y(t) = \lambda y(t) \tag{1}$$

with Dirichlet boundary conditions

$$y(0) = y(1) = 0 \tag{2}$$

for $q(t) \in C[0,1]$ and $t \in [0,1]$.

Pythagoras
BC 570-495

Concerning numerical solution of the Sturm-Liouville problems, finite difference methods are very popular.

Generally speaking, finite difference methods (including asymptotic correction techniques, (Anderssen&De Hoog)[1], (Andrew)[2], extrapolation, (Somali&Oger)[3] have the advantage of simplicity and programming ease.

But it is inefficient for computation of higher eigenvalues.
Asymptotic correction has proved most successful when the derivatives of $q(t)$ are small.

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[1] Anderssen,R.S.,& De Hoog,F.R.(1984). On the correction of finite difference eigenvalue approximations for Sturm-Liouville problems with general boundary conditions. BIT Numerical Mathematics,24(4),401-412.

[2] Andrew,A.L.(1988)Correction of finite element eigenvalues for problems with natural or periodic boundary conditions. BIT Numerical Mathematics, 28(2), 254-269. 2

[3] Somali,S.,&Oger,V.(2005).Improvement of eigenvalues of Sturm-Liouville problem with t-periodic boundary conditions. Journal of Computational and Applied mathematics, 180(2),433-441

Euclid

BC 330-275

The Sequential Splitting Method for Cauchy Problem

The main idea of the splitting method is to lead the complex problem to the sequence of sub-problems with simpler structure. (Geiser)[4]

$$\frac{dY(t)}{dt} = (A+B)\,Y(t) \qquad t \in [0,T] \quad \text{with} \quad Y(0) = Y_0\,, \qquad (3)$$

where $A, B \in \mathbb{R}^{m \times m}$ are constant matrices, $Y = (y_1, \ldots, y_m)^T$ is the solution vector, the initial condition $Y_0 \in \mathbb{R}^m$ is a given constant vector.

The solution is given as

$$Y(t) = e^{t(A+B)} Y_0.$$

[4] Geiser,J.(2011) Iterative splitting methods for differential equations. CRC Press.

Ömer
1048-1131

The method solves two subproblems sequentially an subintervals $[t_i, t_{i+1}]$, for $i = 0, 1, \ldots, N-1$,

$$\frac{dU(t)}{dt} = A\,U(t) \quad \text{with} \quad U(t_i) = Y_{sp,i} \tag{4}$$

$$\frac{dV(t)}{dt} = B\,V(t) \quad \text{with} \quad V(t_i) = U(t_{i+1})\,, \tag{5}$$

where $Y_{sp,0} = Y_0$ and $Y_{sp,i+1} = V(t_{i+1})$, $t_0 = 0$ and $t_N = T$.

Fibonacci
1170-1250

The exact solutions of the equation (4) and (5) respectively are

$$U(t_{i+1}) = e^{(t_{i+1}-t_i)A} Y_{sp,i}$$

and

$$\begin{aligned} V(t_{i+1}) &= e^{(t_{i+1}-t_i)B} U(t_{i+1}) \\ &= e^{(t_{i+1}-t_i)B} e^{(t_{i+1}-t_i)A} Y_{sp,i} \end{aligned}$$

The approximate split solution at the point t_{i+1} is defined as $Y_{sp,i+1} = V(t_{i+1})$. That is

$$Y_{sp,i+1} = e^{hB} e^{hA} Y_{sp,i} \ ,$$

where $h = t_{i+1} - t_i$ is the stepsize.


Galileo
1564–1642

The local splitting error of the sequential splitting method is obtained as

$$\text{Err}_{local} = (e^{h(A+B)} - e^{hB}e^{hA})Y_{sp,i}$$
$$= \frac{1}{2}h^2\,(BA - AB)\,Y_{sp,i} \;+\; \mathcal{O}(h^3)$$

and then the global error of the method

$$\text{Err}_{global} = \;\mathcal{O}(h)$$

when $AB \neq BA$. The splitting error is $O(h)$. So, it is called **First-Order Splitting Method**

Descartes
1596-1650

The Symmetrical Weighted Sequential Splitting Method

We consider the *Cauchy Problem* (3) and define the splitting of the operator on the time interval $[t_i, t_{i+1}]$ as the following

$$\frac{dU_1(t)}{dt} = A\,U_1(t) \quad \text{with} \quad U_1(t_i) = Y_{sp,i}$$
$$\frac{dV_1(t)}{dt} = B\,V_1(t) \quad \text{with} \quad V_1(t_i) = U_1(t_{i+1})$$

and

$$\frac{dU_2(t)}{dt} = B\,U_2(t) \quad \text{with} \quad U_2(t_i) = Y_{sp,i}$$
$$\frac{dV_2(t)}{dt} = A\,V_2(t) \quad \text{with} \quad V_2(t_i) = U_2(t_{i+1})\ ,$$

where $Y_{sp,0} = Y_0$.

Fermat
1601-1665

The approximate split solution at the point $t_{i+1} = t_i + h$ is defined as

$$Y_{sp,i+1} = \frac{1}{2}\{V_1(t_{i+1}) + V_2(t_{i+1})\} \tag{6}$$

$$= \frac{1}{2}\{e^{hB}e^{hA} + e^{hA}e^{hB}\}Y_{sp,i} \; .$$

Pascal
1623-1662

The local spliting error of the symmetrical weighted splitting method is

$$\text{Err}_{local} = \left(e^{h(A+B)} - \frac{1}{2}\left[e^{hB}e^{hA} + e^{hA}e^{hB} \right] \right) Y_{sp,i} \; , \\ = \; \mathcal{O}(h^3) \; ,$$

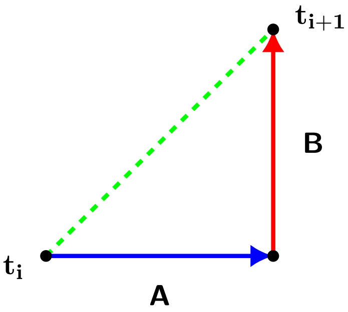
and

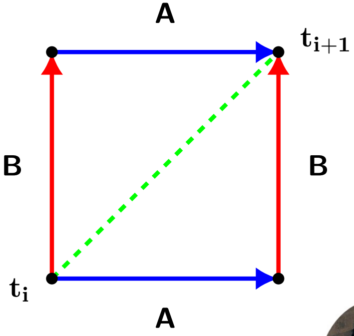
$$\text{Err}_{global} = \; \mathcal{O}(h^2) \; ,$$

The splitting error is $O(h^2)$ if $AB \neq BA$. So, it is called **Second-Order Splitting Method**

Newton
1643-1727

The diagram of splitting methods





Leibniz
1646-1716

Application The Symmetrical Weighted Sequential Splitting Method To Regular SLP

Sturm-Liouville problem (1) and (2) are equivalent with the first order system by $y' = z$

$$Y'(t) = A(t)Y(t) \quad , \quad 0 \le t \le 1 \, , \tag{7}$$

$$C_1 Y(0) + C_2 Y(1) = \mathbf{0} \, , \tag{8}$$

Bernoulli
1655-1705

Application The Symmetrical Weighted Sequential Splitting Method To Regular SLP

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$$Y'(t) = A(t)Y(t) \quad , \quad 0 \le t \le 1 \, , \tag{7}$$

$$C_1 Y(0) + C_2 Y(1) = \mathbf{0} \, , \tag{8}$$

where

$$Y(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} \quad , \quad A(t) = \begin{bmatrix} 0 & 1 \\ q(t) - \lambda & 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Bernoulli
1655-1705

The matrix $A(t)$ is splitted as a sum of M and $q(t)N$

$$A(t) = M + q(t)N,$$

where

$$M = \begin{bmatrix} 0 & 1 \\ -\lambda & 0 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We consider the partition of the interval $[0, 1]$

$$t_i = ih \quad , \quad i = 0, 1, \dots, n \quad , \quad h = \frac{1}{n} \; .$$

L'Hôpital
1661-1704

The symmetrical weighted sequential splitting of the system on time interval $[t_i, t_{i+1}]$ is defined as in the following algorithm,

$$\begin{aligned} U_1'(t) &= M\ U_1(t) & U_1(t_i) &= Y_{sp,i} \\ V_1'(t) &= q(t)N\ V_1(t) & V_1(t_i) &= U_1(t_{i+1}) \end{aligned}$$

and

$$\begin{aligned} U_2'(t) &= q(t)N\ U_2(t) & U_2(t_i) &= Y_{sp,i} \\ V_2'(t) &= M\ V_2(t) & V_2(t_i) &= U_2(t_{i+1}), \end{aligned}$$

for $i = 0, 1, \ldots, n-1$ and $Y_{sp,0}$ is a vector to be determined.


Taylor
1685-1731

The approximate split solution at the point t_{i+1} is defined as

$$Y_{sp,i+1} = \frac{1}{2}\left\{V_1(t_{i+1}) + V_2(t_{i+1})\right\} \ ,$$

$$= \frac{1}{2}\left\{e^{s_{i+1}N}e^{hM} + e^{hM}e^{s_{i+1}N}\right\} Y_{sp,i} \ ,$$

where $s_{i+1} = \int_{t_i}^{t_{i+1}} q(\xi) d\xi$, $i = 0, 1, \ldots, n-1$.

Maclaurin
1698-1746

Finally, we can write the approximate split solution of (7) at $t_n = 1$ as

$$Y_{sp,n} = KY_{sp,0} \approx Y(1) \ ,$$

where K is 2×2 matrix

$$K = \frac{1}{2^n} \left\{ \prod_{i=0}^{n-1} [e^{s_{n-i}N} e^{hM} + e^{hM} e^{s_{n-i}N}] \right\} .$$

Cramer
1704-1752

It is apparent that

$$M^{2j} = (-1)^j \lambda^j I \ , \tag{9}$$

$$M^{2j+1} = (-1)^j \lambda^j M \qquad \text{for} \quad j = 0, 1, \ldots \tag{10}$$

Using (9) and (10), we have

$$e^{tM} = \cos(\sqrt{\lambda} t) I_{2\times 2} + \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda} t) M$$

$$= \begin{bmatrix} \cos(\sqrt{\lambda} t) & \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda} t) \\ -\sqrt{\lambda} \sin(\sqrt{\lambda} t) & \cos(\sqrt{\lambda} t) \end{bmatrix}.$$

Emilie

Since N is nilpotent matrix of index 2 ($N^k = 0\,,\ k \geq 2$), it is clear that

$$e^{s_{n-i}N} = I + s_{n-i}N\ . \tag{11}$$

We obtained that

$$K = \frac{1}{2^n}\left\{\prod_{i=0}^{n-1}[2e^{hM} + s_{n-i}[b(\lambda)I + 2a(\lambda)N]]\right\}\ .$$

where

$$a(\lambda) = \cos(\sqrt{\lambda}h) \quad \text{and} \quad b(\lambda) = \frac{\sin(\sqrt{\lambda}h)}{\sqrt{\lambda}}\ .$$

Euler
1707-1783

The solution $Y_{sp,n}$ will be the solution of (7) and (8)

$$C_1 Y_{sp,0} + C_2 Y_{sp,n} = \mathbf{0}$$
$$(C_1 + C_2 K) Y_{sp,0} = \mathbf{0} \ .$$

For a non-trivial solution $Y_{sp,0}$, the determinant of $C_1 + C_2 K$ must be zero. It follows that

$$Q(\lambda) = \det(C_1 + C_2 K)$$

is the approximate characteristic function of SLP (7).
Note that; $Q(\lambda)$ is the $(1,2)^{th}$ entry of K.

D'Alembert
1717-1783

If $q(t) = 0$, then $s_i = 0$. Since $nh = 1$, we have

$$K = \frac{1}{2^n} \prod_{i=0}^{n-1} 2e^{hM} = e^M.$$

From $\det(C_1 + C_2 K) = 0$, we get the characteristic equation of the original SLP

$$\frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}) = 0$$

and then the eigenvalues of SLP (1) and (2) are $\lambda_k = k^2\pi^2$, $k = 1, 2, \ldots$.

Maria
1718-1799

Now, we consider the case $q(t)$ is constant that is $q(t) = q$, then K will be

$$K = \frac{1}{2^n}[2e^{hM} + qh(bI + 2aN)]^n$$

$$= \frac{1}{2^n}L^n,$$

where $L = \begin{bmatrix} 2a + qhb & 2b \\ -2\lambda b + 2aqh & 2a + qhb \end{bmatrix}$ and $a(\lambda) := a$, $b(\lambda) := b$ for simplicity.

Laplace
1749-1827

From the determinant of matrix $(C_1 + \frac{1}{2^n} C_2 L^n)$, we have the characteristic function $Q(\lambda) \in \mathbb{R}$ as the following

$$Q(\lambda) = \frac{-1}{2^{n+1}} \frac{b\sqrt{n}}{\sqrt{aqb - b^2 n\lambda}} (\mu_2^n - \mu_1^n), \qquad (12)$$

where

$$\mu_1^n = \left(\frac{1}{n}\right)^n [2an + qb + 2\sqrt{bn(-\lambda bn + aq)}]^n$$

and

$$\mu_2^n = \left(\frac{1}{n}\right)^n [2an + qb - 2\sqrt{bn(-\lambda bn + aq)}]^n.$$

Legendre
1752-1833

We get limit of the characteristic equation $Q(\lambda)$ as

$$\lim_{n\to\infty} Q(\lambda) = \frac{1}{\sqrt{\lambda - q}} \left\{ \frac{e^{i\sqrt{\lambda-q}} - e^{-i\sqrt{\lambda-q}}}{2i} \right\},$$

$$= \frac{1}{\sqrt{\lambda - q}} \sin\sqrt{\lambda - q},$$

where $\lambda - q > 0$.

Fourier
1768-1830

Asymptotic Behaviour for Eigenvalues of SLP

In order to derive the error estimate $e_s = \wedge_s - \lambda_s^{(p+1)}$, it is necessary to examine in some details of the asymptotic behaviour of e_s for constant case $q(t) = q$. Let

$$|e_s| = | \wedge_s - \lambda_s^{(p+1)}| = \left| \wedge_s - \left\{ \lambda_s^{(p)} - F(\lambda_s^{(p)}) \right\} \right|,$$

where $\lambda_s^{(p)}$ is the s^{th} approximate eigenvalue to the s^{th} eigenvalues $\wedge_s$ of the original SLP that obtained by Newton method at p^{th} step, $F(\lambda)$ is the reduced rational function to $\frac{Q(\lambda)}{Q'(\lambda)}$ such that $F(\lambda_s^{(p)})$ is defined, $Q(\lambda)$ in (12) is approximate characteristic equation that obtained from the symmetrical weighted splitting method.

Sophie
1776-1831

$Q(\lambda)$ is zero whenever λ is an eigenvalue depending on n (number of intervals), but it is also zero when $\lambda = n^2k^2\pi^2$, $k = 1, 2, \ldots,$ which are not eigenvalues for $q(t) = q$.

Therefore, the removing these extraneous zeros, we will discuss the error formula in two cases.

Gauss
1777-1855

Case i : Let $s = nk + j$, $\lambda_s^{(0)} = (nk+j)^2\pi^2$ and $j = \frac{n}{2}$, n is even number of interval, then

$$|e_s| = |e_{\frac{n}{2}(2k+1)}| \leq \frac{|c_1|}{\lambda_s^{(0)}}, \tag{13}$$

where $c_1 = (q^2 - \frac{1}{12}q^3) + \mathcal{O}(\frac{1}{n})$,

$$s > \frac{\sqrt{|q^2 - \frac{1}{12}q^3|}}{\pi}, \tag{14}$$

for any even $n \geq 2$.

Mary
1780-1872

Case ii : Let $s = nk + j$, $\lambda_s^{(0)} = (nk+j)^2\pi^2$ and $j \neq \frac{n}{2}$, we get

$$|e_s| = |e_{nk+j}| \leq \frac{|d_1|}{\sqrt{\lambda_s^{(0)}}}, \tag{15}$$

where

$$d_1 = \frac{\cos^3(\frac{j}{n}\pi)q^2}{4n\sin(\frac{j}{n}\pi)} + \mathcal{O}(\frac{1}{n^2}),$$

$$s > \frac{q^2}{4\pi^2}. \tag{16}$$

Cauchy
1789-1857

As a result, from the asymptotic expansion of the error formula, we obtain that

$$|\wedge_s - \lambda_s^{(p+1)}| = \begin{cases} \mathcal{O}(\frac{1}{s^2}), & s = \frac{n}{2}(2k+1), \quad n : \text{even}, \\ \\ \mathcal{O}(\frac{1}{s}), & s = nk + j, \quad j \neq \frac{n}{2}, \end{cases} \quad (17)$$

satisfying the conditions (14) and (16) corresponding to the choosen n.

Galois
1811-1832

For the constant case $q(t) = q$, we use forward difference technique to correct the eigenvalues using the property,

$$\Delta^3 \wedge_k = 0.$$

Suppose that for $s + 4$ values,

$$\lambda_k = \wedge_k + \delta, \quad k = s+1, \ldots, s+4,$$

where δ is sufficiently small and

$$\lambda_k = \wedge_k + \epsilon_k, \quad k = 1, 2, \ldots, s,$$

where ϵ_k is the error for each k.

Ada

Using the forward difference formula, we obtain that

$$\begin{aligned}
\Delta^3\lambda_s &= -\epsilon_s + \delta \approx \epsilon_s \\
\Delta^3\lambda_{s-1} &= 2\epsilon_s + \Delta\epsilon_{s-1} \\
\Delta^3\lambda_{s-2} &= -\epsilon_s - \Delta\epsilon_{s-1} - \Delta^2\epsilon_{s-2} \\
\Delta^3\lambda_k &= \Delta^3\epsilon_k, \qquad k = 1, 2, \ldots, s-3.
\end{aligned}$$

Solving all errors from ϵ_s to ϵ_1, we correct the first k eigenvalues $\lambda_k^{(c)}$ with the accuracy δ of $\wedge_r$ for $r \geq s+1$, in the following formula

$$\lambda_k^{(c)} = \lambda_k - \epsilon_k, \quad k = 1, \ldots, s.$$

Weierstrass
1815-1897

Numerical Results

For the numerical results, the observed orders are obtained the following formulas

$$order = \log\left(\frac{\wedge_s - \lambda_{s,n}}{\wedge_r - \lambda_{r,n}}\right) / \log\left(\frac{r}{s}\right) \tag{18}$$

or

$$order = \log\left(\frac{\lambda_{s,n} - \lambda_{s,m}}{\lambda_{r,n} - \lambda_{r,m}}\right) / \log\left(\frac{r}{s}\right), \tag{19}$$

where $\lambda_{s,n}$ and $\lambda_{s,m}$ are the approximate eigenvalues to $\wedge_s$ for n, m respectively.

Riemann
1826-1866

Comparison of the eigenvalues

For $n = 2$, $j = 1$ and $n = 6$, $j = 3$ with $q(t) = 2$.			
s	$\lvert\lambda_{s,2} - \wedge_s\rvert$	$\lvert\lambda_{s,6} - \wedge_s\rvert$	$order$
3	1.28236E-2		
15	5.24858E-4		
63	2.97812E-5		
141	5.94571E-6		
219	2.46457E-6		
321	1.14716E-6		
411	6.99656E-7		
501	4.70784E-7		

$\wedge_s$ The s^{th} exact eigenvalue.

$\lambda_{s,n}$ The computed s^{th} approximate eigenvalue for choosen n.

Lipschitz
1832-1903

Comparison of the eigenvalues

For $n = 2$, $j = 1$ and $n = 6$, $j = 3$ with $q(t) = 2$.			
s	$\lvert \lambda_{s,2} - \wedge_s \rvert$	$\lvert \lambda_{s,6} - \wedge_s \rvert$	$order$
3	1.28236E-2	1.12130E-2	
15	5.24858E-4	4.58239E-4	
63	2.97812E-5	2.59995E-5	
141	5.94571E-6	5.19070E-6	
219	2.46457E-6	2.15159E-6	
321	1.14716E-6	1.00129E-6	
411	6.99656E-7	6.10249E-7	
501	4.70784E-7	4.11179E-7	

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Lipschitz
1832-1903

Comparison of the eigenvalues

For $n = 2$, $j = 1$ and $n = 6$, $j = 3$ with $q(t) = 2$.			
s	$\|\lambda_{s,2} - \wedge_s\|$	$\|\lambda_{s,6} - \wedge_s\|$	$order$
3	1.28236E-2	1.12130E-2	-1.97920
15	5.24858E-4	4.58239E-4	-1.99793
63	2.97812E-5	2.59995E-5	-1.99986
141	5.94571E-6	5.19070E-6	-1.99991
219	2.46457E-6	2.15159E-6	-1.99661
321	1.14716E-6	1.00129E-6	-1.98059
411	6.99656E-7	6.10249E-7	-2.04767
501	4.70784E-7	4.11179E-7	-2.05363

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$\lambda_{s,n}$ The computed s^{th} approximate eigenvalue for choosen n.

Lipschitz
1832-1903

Comparison of the eigenvalues

For $n = 2$, $j = 1$ and $n = 6$, $j = 3$ with $q(t) = 5$.			
s	$\lvert\lambda_{s,2} - \wedge_s\rvert$	$\lvert\lambda_{s,6} - \wedge_s\rvert$	$order$
3	9.34553E-2		
15	3.97642E-3		
63	2.25996E-4		
141	1.36722E-4		
219	1.87049E-5		
321	8.70635E-6		
411	5.31063E-6		
501	3.57348E-6		

$\wedge_s$ The s^{th} exact eigenvalue.

$\lambda_{s,n}$ The computed s^{th} approximate eigenvalue for choosen n.

Schwarz
1843-1921

Comparison of the eigenvalues

For $n = 2$, $j = 1$ and $n = 6$, $j = 3$ with $q(t) = 5$.			
s	$\|\lambda_{s,2} - \wedge_s\|$	$\|\lambda_{s,6} - \wedge_s\|$	$order$
3	9.34553E-2	6.96624E-2	
15	3.97642E-3	2.93801E-3	
63	2.25996E-4	1.66916E-4	
141	1.36722E-4	3.33262E-5	
219	1.87049E-5	1.38147E-5	
321	8.70635E-6	6.43008E-6	
411	5.31063E-6	3.92250E-6	
501	3.57348E-6	2.63983E-6	

$\wedge_s$ The s^{th} exact eigenvalue.

$\lambda_{s,n}$ The computed s^{th} approximate eigenvalue for choosen n.

Schwarz
1843-1921

Comparison of the eigenvalues

For $n = 2,\ j = 1$ and $n = 6,\ j = 3$ with $q(t) = 5$.			
s	$\lvert\lambda_{s,2} - \wedge_s\rvert$	$\lvert\lambda_{s,6} - \wedge_s\rvert$	$order$
3	9.34553E-2	6.96624E-2	-1.94583
15	3.97642E-3	2.93801E-3	-1.99442
63	2.25996E-4	1.66916E-4	-1.99966
141	1.36722E-4	3.33262E-5	-1.99992
219	1.87049E-5	1.38147E-5	-1.99989
321	8.70635E-6	6.43008E-6	-2.00110
411	5.31063E-6	3.92250E-6	-2.00297
501	3.57348E-6	2.63983E-6	-2.00277

$\wedge_s$ The s^{th} exact eigenvalue.

$\lambda_{s,n}$ The computed s^{th} approximate eigenvalue for choosen n.

Schwarz
1843-1921

Comparison of the eigenvalues

For $n = 3,\ j = 1$ and $n = 5,\ j = 1$ with $q(t) = 2$.

s	$\lvert\lambda_{s,3} - \wedge_s\rvert$	$\lvert\lambda_{s,5} - \wedge_s\rvert$	$order$
1	8.18589E-2		
16	5.09730E-4		
61	2.20290E-4		
121	1.18749E-4		
211	7.00104E-5		
301	4.96161E-5		
436	3.45239E-5		
541	2.79178E-5		

$\wedge_s$ The s^{th} exact eigenvalue.

$\lambda_{s,n}$ The computed s^{th} approximate eigenvalue for choosen n.

Christine

1847-1930

Comparison of the eigenvalues

For $n = 3,\ j = 1$ **and** $n = 5,\ j = 1$ **with** $q(t) = 2$.			
s	$\lvert\lambda_{s,3} - \wedge_s\rvert$	$\lvert\lambda_{s,5} - \wedge_s\rvert$	$order$
1	8.18589E-2	4.30804E-2	
16	5.09730E-4	3.15470E-3	
61	2.20290E-4	9.10733E-4	
121	1.18749E-4	4.66494E-4	
211	7.00104E-5	2.69345E-4	
301	4.96161E-5	1.89325E-4	
436	3.45239E-5	1.30962E-4	
541	2.79178E-5	1.05634E-4	

$\wedge_s$ The s^{th} exact eigenvalue.

$\lambda_{s,n}$ The computed s^{th} approximate eigenvalue for choosen n.

Christine
1847-1930

Comparison of the eigenvalues

For $n = 3$, $j = 1$ and $n = 5$, $j = 1$ with $q(t) = 2$.

s	$\|\lambda_{s,3} - \wedge_s\|$	$\|\lambda_{s,5} - \wedge_s\|$	$order$
1	8.18589E-2	4.30804E-2	-0.96848
16	5.09730E-4	3.15470E-3	-1.00459
61	2.20290E-4	9.10733E-4	-1.00170
121	1.18749E-4	4.66494E-4	-1.00076
211	7.00104E-5	2.69345E-4	-1.00048
301	4.96161E-5	1.89325E-4	-1.00034
436	3.45239E-5	1.30962E-4	-1.00026
541	2.79178E-5	1.05634E-4	-1.00023

$\wedge_s$ The s^{th} exact eigenvalue.

$\lambda_{s,n}$ The computed s^{th} approximate eigenvalue for choosen n.

Christine
1847-1930

Comparison of the eigenvalues

For $n = 3$, $j = 1$ and $n = 5$, $j = 1$ with $q(t) = 5$.			
s	$\lvert\lambda_{s,3} - \wedge_s\rvert$	$\lvert\lambda_{s,5} - \wedge_s\rvert$	$order$
1	4.76135E-1		
16	2.70402E-3		
61	1.34364E-3		
121	7.33757E-4		
211	4.34797E-4		
301	3.08740E-4		
436	2.15123E-4		
541	1.74061E-4		

$\wedge_s$ The s^{th} exact eigenvalue.

$\lambda_{s,n}$ The computed s^{th} approximate eigenvalue for choosen n.

Charlotte 1858-1931

Comparison of the eigenvalues

For $n = 3$, $j = 1$ and $n = 5$, $j = 1$ with $q(t) = 5$.			
s	$\lvert\lambda_{s,3} - \wedge_s\rvert$	$\lvert\lambda_{s,5} - \wedge_s\rvert$	$order$
1	4.76135E-1	2.61196E-1	
16	2.70402E-3	1.93971E-2	
61	1.34364E-3	5.67137E-3	
121	7.33757E-4	2.91039E-3	
211	4.34797E-4	1.68170E-3	
301	3.08740E-4	1.18245E-3	
436	2.15123E-4	8.18115E-4	
541	1.74061E-4	6.59956E-4	

$\wedge_s$ The s^{th} exact eigenvalue.

$\lambda_{s,n}$ The computed s^{th} approximate eigenvalue for choosen n.

Charlotte
1858-1931

Comparison of the eigenvalues

For $n = 3,\ j = 1$ **and** $n = 5,\ j = 1$ **with** $q(t) = 5$**.**			
s	$\lvert\lambda_{s,3} - \wedge_s\rvert$	$\lvert\lambda_{s,5} - \wedge_s\rvert$	$order$
1	4.76135E-1	2.61196E-1	-0.92165
16	2.70402E-3	1.93971E-2	-1.01109
61	1.34364E-3	5.67137E-3	-1.00420
121	7.33757E-4	2.91039E-3	-1.00188
211	4.34797E-4	1.68170E-3	-1.00120
301	3.08740E-4	1.18245E-3	-1.00083
436	2.15123E-4	8.18115E-4	-1.00063
541	1.74061E-4	6.59956E-4	-1.00051

$\wedge_s$ The s^{th} exact eigenvalue.

$\lambda_{s,n}$ The computed s^{th} approximate eigenvalue for choosen n.

Charlotte
1858-1931

Correction of the errors of the eigenvalues

For $n = 2$, $j = 1$ and $n = 6$, $j = 3$ with $q(t) = 2$.

s	$\lvert\lambda_{s,2} - \wedge_s\rvert$	$\lvert\lambda_{s,2}^{(c)} - \wedge_s\rvert$	$\lvert\lambda_{s,6} - \wedge_s\rvert$	$\lvert\lambda_{s,6}^{(c)} - \wedge_s\rvert$
3	1.2824E-2	5.3594E-5		
9	1.4554E-3	5.2534E-5		
15	5.2485E-4	5.1483E-5		
21	2.6791E-4	5.0444E-5		
27	1.6210E-4	4.9415E-5		
33	1.0852E-4	4.8397E-5		
39	7.7706E-5	4.7390E-5		
45	5.8368E-5	4.6393E-5		

$\wedge_s$ The s^{th} exact eigenvalue.

$\lambda_{s,n}$ The computed s^{th} approximate eigenvalue for choosen n.

$\lambda_{s,n}^{(c)}$ The corrected eigenvalue obtained from forward difference technique

Sonja
1850-1891

Correction of the errors of the eigenvalues

For $n = 2$, $j = 1$ and $n = 6$, $j = 3$ with $q(t) = 2$.

s	$\|\lambda_{s,2} - \wedge_s\|$	$\|\lambda_{s,2}^{(c)} - \wedge_s\|$	$\|\lambda_{s,6} - \wedge_s\|$	$\|\lambda_{s,6}^{(c)} - \wedge_s\|$
3	1.2824E-2	5.3594E-5	1.1213E-2	1.8836E-5
9	1.4554E-3	5.2534E-5	1.2708E-3	1.8710E-5
15	5.2485E-4	5.1483E-5	4.5824E-4	1.8584E-5
21	2.6791E-4	5.0444E-5	2.3390E-4	1.8459E-5
27	1.6210E-4	4.9415E-5	1.4152E-4	1.8334E-5
33	1.0852E-4	4.8397E-5	9.4745E-5	1.8210E-5
39	7.7706E-5	4.7390E-5	6.7839E-5	1.8085E-5
45	5.8368E-5	4.6393E-5	5.0956E-5	1.7962E-5

$\wedge_s$ The s^{th} exact eigenvalue.

$\lambda_{s,n}$ The computed s^{th} approximate eigenvalue for choosen n.

$\lambda_{s,n}^{(c)}$ The corrected eigenvalue obtained from forward difference technique

Sonja
1850-1891

Finite Difference Method

For $n = 2$ with $q(t) = 2$.			
s	$\wedge_s$	$\lvert \wedge_s - \lambda^{(f)}_{s,20} \rvert$	$\lvert \wedge_s - \lambda_{s,2} \rvert$
1	11.8696044	2.0277E-2	9.7745E-2
3	90.8264396	1.6317	1.2824E-2
5	248.740110	12.4255	4.6873E-3
7	485.610615	46.8030	2.4017E-3
9	801.437956	124.5855	1.4554E-3
11	1196.22213	269.0746	9.7516E-4
13	1669.96314	504.7707	6.9855E-4
15	2222.66099	854.9756	5.2486E-4

$\wedge_s$ The s^{th} exact eigenvalue.

$\lambda_{s,n}$ The computed s^{th} approximate eigenvalue for choosen n.

$\lambda^{(f)}_{s,n}$ The eigenvalue obtained from finite difference approximation for choos[en] n.

Hilbert
1862-1943

$-y''(t) + e^t y(t) = \lambda y(t), \qquad y(0) = y(1) = 0$				
s	n	λ_s^*	$\lvert\lambda_{s,39}^{(f)} - \lambda^*\rvert$	$\lvert\lambda_{s,n} - \lambda_s^*\rvert$
1	6	11.5424	0.0057	0.1543E-1
2	4	41.1867	0.0813	0.8668E-2
3	6	90.5404	0.4106	0.3988E-2
4	6	159.6296	1.2954	0.7742E-2
5	2	248.4569	3.1544	0.1902E-2
6	4	357.023	6.5261	0.1114E-2
7	2	485.3281	12.0593	0.9407E-3
8	5	633.3724	20.5083	0.2615E-2
9	6	801.1558	32.7373	0.5008E-3
10	4	988.6783	49.7023	0.3562E-3

λ_s^* The eigenvalues are in (Paine, de Hoog,& Anderssen)[5].

$\lambda_{s,n}^{(f)}$ The eigenvalue obtained from finite difference approximation for choosen n.


Cahit Arf 1910-1997

[5] Paine, J. W., de Hoog, F.R.& Anderssen, R. S.(1981). On the correction of finite difference eigenvalue approximations for Sturm-Liouville problems. Computing,26(2), 123-139

The greater than ten eigenvalues

For $n = 2$, $j = 1$ **and** $n = 6$, $j = 3$ **with** $q(t) = e^t$.

s	$\lambda_{s,2}$	$\lvert\lambda_{s,2} - \lambda_{s,6}\rvert$	$order$
15	2222.3788924	4.10845E-5	-1.99823
21	4354.2136289	2.09740E-5	-1.99936
45	19987.667151	4.56988E-6	-1.99981
69	46990.904817	1.94387E-6	-1.99997
87	74704.753982	1.22272E-6	-2.
129	164241.80511	5.56145E-7	-2.00039
237	554367.52788	1.64728E-7	-2.00442
351	1215946.8500	7.49715E-8	-1.99589
405	1618863.5801	5.63450E-8	-1.91204
513	2597375.6389	3.58559E-8	-2.20865

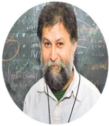

John Nash
1928-2015

$$-y''(t) + t^2 y(t) = \lambda y(t), \qquad y(0) = y(1) = 0$$

s	n	λ_s^*	$\lvert\lambda_{s,20}^{(f)} - \lambda_s^*\rvert$	$\lvert\lambda_{s,n} - \lambda_s^*\rvert$
1	7	10.1511643	2.0291E-2	5.99769E-3
2	7	39.7993930	3.2365E-1	5.39722E-3
3	5	89.1543424	1.6316885	3.00800E-3
4	6	158.243961	5.1273118	1.80503E-3
5	2	247.071500	12.425603	1.82758E-3
6	4	355.637743	25.534059	2.68230E-3
7	2	483.942959	46.803153	9.30714E-4
8	5	631.987257	78.868467	1.39727E-3
9	2	799.770691	124.58579	5.62593E-4
10	7	987.293288	186.96079	7.50294E-5

λ_s^* The eigenvalues are in (Birkhoff & Varga)[6].

$\lambda_{s,n}^{(f)}$ The eigenvalue obtained from finite difference approximation for choosen n.



Ali Nesin
1957-

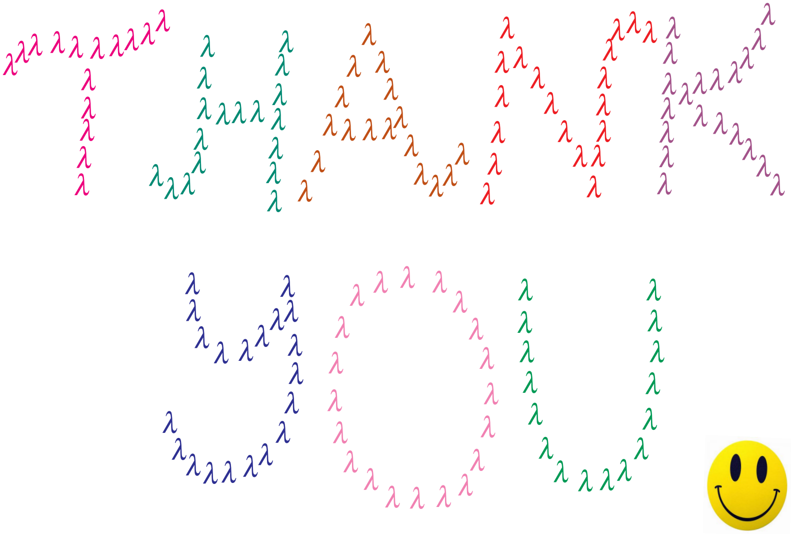
[6] Birkhoff, G., & Varga, R. S. (1970). Numerical solution of field problems in continuum physics, volume 2. Rhode Island: American Mathematical Society

The greater than ten eigenvalues

For $n=2,\ j=1$ **and** $n=6,\ j=3$ **with** $q(t)=t^2$.

s	$\lambda_{s,2}$	$\lvert\lambda_{s,2}-\lambda_{s,6}\rvert$	$order$
21	4352.8288676	1.08987E-7	-1.99972
27	7195.2749377	6.59347E-8	-2.00004
33	10748.332523	4.41378E-8	-1.99971
45	19986.282244	2.37414E-8	-2.00125
51	25671.174379	1.84809E-8	-2.00180
63	39172.793200	1.21217E-8	-2.00679
81	64754.807808	7.34872E-9	-1.99191
87	74703.369044	6.37374E-9	-2.01201
105	108812.72185	4.36557E-9	-2.02980
147	213272.61483	2.24099E-9	-2.02787

Maryem 1977-


THANK
YOU

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