③ Show that $\lim\limits_{x \to \sqrt{3}} \frac{1}{x^2} = \frac{1}{3}$

Solution: $x_0 = \sqrt{3}$ $\quad f(x) = \frac{1}{x^2}$ $\quad L = \frac{1}{3}$

$|x - \sqrt{3}| < \delta \Rightarrow |f(x) - \frac{1}{3}| < \varepsilon$. Find δ ?

Step 1

$$|f(x) - \frac{1}{3}| = |\frac{1}{x^2} - \frac{1}{3}| < \varepsilon \Rightarrow \frac{1}{3} - \varepsilon < \frac{1}{x^2} < \frac{1}{3} + \varepsilon$$

$$\Rightarrow \frac{3}{1+3\varepsilon} < x^2 < \frac{3}{1-3\varepsilon}$$

$$\Rightarrow \sqrt{\frac{3}{1+3\varepsilon}} < |x| < \sqrt{\frac{3}{1-3\varepsilon}}$$

For x near $\sqrt{3}$, $\sqrt{\frac{3}{1+3\varepsilon}} < x < \sqrt{\frac{3}{1-3\varepsilon}}$.

Step 2

$$|x - \sqrt{3}| < \delta \Rightarrow \sqrt{3} - \delta < x < \sqrt{3} + \delta$$

Then $\sqrt{3} - \delta = \sqrt{\frac{3}{1+3\varepsilon}}$ $\quad$ or $\quad \sqrt{3} + \delta = \sqrt{\frac{3}{1-3\varepsilon}}$

$\delta = \sqrt{3} - \sqrt{\frac{3}{1+3\varepsilon}}$ $\qquad \delta = \sqrt{\frac{3}{1-3\varepsilon}} - \sqrt{3}$

Choose $\delta = \min\left\{ \sqrt{3} - \sqrt{\frac{3}{1+3\varepsilon}} , \sqrt{\frac{3}{1-3\varepsilon}} - \sqrt{3} \right\}$.

Find the asymptotes of the function

$$f(x) = x + \sqrt{x^2-1}$$

Solution: Examine the domain of the function

$$x^2-1 \geq 0 \quad \Rightarrow \quad x \in (-\infty, -1] \cup [1, \infty)$$

Vertical Asymptote: Check for vertical asymptotes at the boundary points $x = \mp 1$

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} x + \sqrt{x^2-1} = -1 \quad \Big| \quad \lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} x + \sqrt{x^2-1} = -1$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x + \sqrt{x^2-1} = 1 \quad \Big| \quad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x + \sqrt{x^2-1} = 1$$

Thus, the function has no vertical asymptotes.

Horizontal Asymptotes: Now we look for horizontal asymptote

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} x + \sqrt{x^2-1} = \infty$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x + \sqrt{x^2-1} = \infty - \infty \quad (\text{indeterminate form})$$

To solve this limit we rationalize the expression.

$$\lim_{x \to -\infty} \frac{(x+\sqrt{x^2-1}).(x-\sqrt{x^2-1})}{(x-\sqrt{x^2-1})} = \lim_{x \to -\infty} \frac{x^2-(\sqrt{x^2-1})^2}{x-\sqrt{x^2-1}} = \lim_{x \to -\infty} \frac{1}{x-\sqrt{x^2-1}} = 0$$

So, we get a horizontal asymptote $y = 0$

Oblique Asymptote: Consider a possible oblique asymptote $y = mx + n$.

$$m = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{x+\sqrt{x^2-1}}{x} = \lim_{x \to \infty} \left(1 + \sqrt{1-\frac{1}{x^2}}\right) = 1 + 1 = 2_{//}$$

$$n = \lim_{x \to \infty} [f(x) - mx] = \lim_{x \to \infty} [x + \sqrt{x^2-1} - 2x] = \lim_{x \to \infty} (-x + \sqrt{x^2-1}) = \lim_{x \to \infty} \frac{(\sqrt{x^2-1})^2 - (x)^2}{\sqrt{x^2-1}+x}$$

$$= \lim_{x \to \infty} \frac{1}{\sqrt{x^2-1}+x} = 0 \quad \Rightarrow \quad y = 2x \text{ is oblique asymptote}_{//}$$

Q: Find the derivative of $\cos(x^2-1)$ using the limit definition.

Solution:

Recall

From trigonometry

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

Fundamental Trigonometric Limits

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \qquad \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$$

$$f(x) = \cos(x^2-1)$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We need to find $\displaystyle \lim_{h \to 0} \left\{ \frac{\cos\left((x+h)^2-1\right) - \cos(x^2-1)}{h} \right\}$

Let's focus on the expression whose limit we need.

$$\frac{\cos\left((x+h)^2-1\right) - \cos(x^2-1)}{h}$$

$$= \frac{\cos\left((x^2-1) + (2xh+h^2)\right) - \cos(x^2-1)}{h}$$

$$= \frac{\cos(x^2-1)\cos(2xh+h^2) - \sin(x^2-1)\sin(2xh+h^2) - \cos(x^2-1)}{h}$$

$$= \cos(x^2-1)\,\frac{\cos(2xh+h^2)-1}{h} - \sin(x^2-1)\,\frac{\sin(2xh+h^2)}{h}$$

$$= \cos(x^2-1) \cdot \underbrace{\frac{\cos(2xh+h^2)-1}{2xh+h^2}}_{\to 0} \cdot (2x+h) - \sin(x^2-1) \cdot \underbrace{\frac{\sin(2xh+h^2)}{2xh+h^2}}_{\to 1} \cdot (2x+h)$$

as $h \to 0$

$$= \cos(x^2-1) \cdot 0 \cdot (2x) - \sin(x^2-1) \cdot (1) \cdot (2x) = -2x\sin(x^2-1) \;//$$

Rec. #4

SOLUTIONS

1) Right hand derivative of $f(x)$ at $x=1$. $\quad f(1) = 0^2 + \sin 0 = 0$

$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{|1+h-1|(1+h)^2 + \sin(1+h-1) - 0}{h}$$

$$= \lim_{h\to 0^+} \frac{h(1+h)^2 + \sin h}{h} = \lim_{h\to 0^+} (1+h)^2 + \frac{\sin h}{h} = 2$$

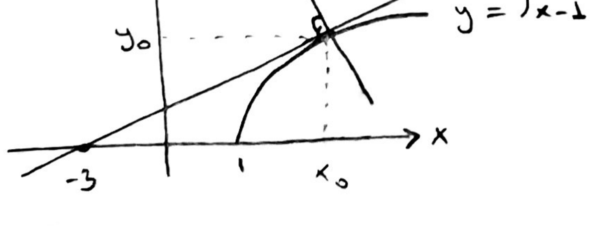
Left hand derivative of $f(x)$ at $x=1$,

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{|1+h-1|(1+h)^2 + \sin(1+h-1) - 0}{h}$$

$$= \lim_{h\to 0^-} \frac{-h(1+h)^2 + \sin h}{h} = \lim_{h\to 0} -(1+h)^2 + \frac{\sin h}{h} = 0$$

Since right and left derivative of $f(x)$ at $x=1$ does not equal, f is <u>not</u> differentiable at $x=1$, $f'(1)$ does not exist.

2) Suppose that the point $P(x_0, y_0)$ where the tangent line to the curve $y=\sqrt{x-1}$ crosses the x-axis at $x=-3$. $P(x_0,y_0)$ is also on the curve, that is $y_0 = \sqrt{x_0 - 1}$.



* We have to find the slope of tangent line (m_T)
* We have to find the equation of tangent line
* We have to find (x_0, y_0)

$$m_T = \left.\frac{df}{dx}\right|_{x=x_0} = \frac{1}{2\sqrt{x_0-1}}, \quad (x_0,y_0) \text{ and } y_0 = \sqrt{x_0-1}.$$

Equation of the tangent line: $\quad y = y_0 + \frac{1}{2\sqrt{x_0-1}} \cdot (x-x_0)$

The tangent line passes through $(-3,0)$, so

$$0 = y_0 + \frac{1}{2\sqrt{x_0-1}}(-3-x_0) \quad , \quad \text{substitue } \sqrt{x_0-1} \text{ on } y_0$$

$$\frac{3+x_0}{2\sqrt{x_0-1}} = \sqrt{x_0-1} \Rightarrow 3+x_0 = 2(x_0-1)$$

$$\Rightarrow x_0 = 5 \Rightarrow y_0 = \sqrt{5-1} = 2$$

$$\Rightarrow m_t = \frac{1}{2\sqrt{5-1}} = \frac{1}{4}$$

Hence, the equation of the tangent line

$$y = 2 + \frac{1}{4}(x-5) \Rightarrow y = \frac{1}{4}x + \frac{3}{4}$$

The slope of the normal line: $m_N = -\frac{1}{m_T}$, so $m_N = -4$

and $P(5,2)$,

The equation of the normal line,

$$y = 2 - 4(x-5) \Rightarrow y = -4x + 22 \,//$$

3) The slope of the horizontal tangent at the point $x = x_0$ is $m = 0$.

$f(x) = x^2 + 4x - 1$

$f'(x) = 2x + 4$

$m = f'(x_0) = 2x_0 + 4$

$2x_0 + 4 = 0 \Rightarrow x_0 = -2$

Hence, the function $f(x)$ has horizontal tangent at $x = -2$ //

4) $\operatorname{cosec}(x^2+y^2) = 1 \Rightarrow \dfrac{1}{\sin(x^2+y^2)} = 1 \Rightarrow \sin(x^2+y^2) = 1$

So, $x^2+y^2 = \frac{\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$

By using implicit differentiation, we get

$$2x + 2.y.y' = 0 \Rightarrow y' = -\frac{x}{y}$$

We may have vertical, tangent $y=0$. Then

$$x^2 + 0^2 = \frac{\pi}{2} + 2k\pi$$

$$x = \mp\sqrt{\frac{\pi}{2} + 2k\pi} \;//$$

5) a. $y = \dfrac{x^3+7}{x} \Rightarrow y = x^2 + 7.x^{-1} \Rightarrow y' = 2x - 7x^{-2}$

b. $y = x^7 + \sqrt{7}x - \dfrac{1}{\pi+1} \Rightarrow y' = 7x^6 + \sqrt{7}$

c. $y = (2x-5)(4-x)^{-1} \Rightarrow y' = (2x-5)'(4-x)^{-1} + (2x-5)[(4-x)^{-1}]'$

Using product rule $\Rightarrow y' = 2(4-x)^{-1} + (2x-5)(-1(4-x)^{-2}(-1))$

$$\Rightarrow y' = \frac{2}{4-x} + \frac{2x-5}{(4-x)^2} = \frac{2(4-x)+2x-5}{(4-x)^2} = \frac{3}{(4-x)^2}$$

Using quotient rule $y = \dfrac{2x-5}{4-x} \Rightarrow y' = \dfrac{(2x-5)'(4-x) - (2x-5)(4-x)'}{(4-x)^2}$

$$y' = \frac{2(4-x) - (2x-5)(-1)}{(4-x)^2} = \frac{3}{(4-x)^2}$$

d. $y = \dfrac{x(x+1)(x^2-x+1)}{x^4} = \dfrac{x^3-1}{x^3} = 1 - x^{-3}$

$$\Rightarrow y' = -(-3)x^{-4} \Rightarrow y' = 3/x^4$$

e. $y=(x^2+1)(x+5+\frac{1}{x})$

$y'=(x^2+1)'(x+5+\frac{1}{x})+(x^2+1)(x+5+\frac{1}{x})'$

$=2x(x+5+\frac{1}{x})+(x^2+1)(1-\frac{1}{x^2})$

$=2x^2+10x+2+\frac{x^4-1}{x^2}=3x^2+10x+2-x^{-2}$ //

f. $y=(\sec x+\tan x)(\sec x-\tan x)=\sec^2 x-\tan^2 x=1$, since $1+\tan^2 x=\sec^2 x$

So, $y=1 \Rightarrow y'=0$

OR $y=\sec^2 x-\tan^2 x$

$y'=2\sec x.\underbrace{\sec x.\tan x}_{(\sec x)'}-2\tan x.\underbrace{\sec^2 x}_{(\tan x)'}=0$ //

g. $y=\tan(x+\cos x)$, $y'=\frac{dy}{dx}=?$

{ Chain rule or outside-inside rule, $y=f(g(x)) \Rightarrow y'=f'(g(x)).g'(x)$ }

Let $u=x+\cos x$, then $y=\tan u$

$u=x+\cos x$

By using chain rule,

$\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}=\sec^2 u.(1-\sin x)$

$\Rightarrow \frac{dy}{dx}=\sec^2(x+\cos x)(1-\sin x)$

h) $y=\tan^2(\sin^3 x)$, $\frac{dy}{dx}=?$

Let $y=u^2$

$u=\tan(v)$

$v=w^3$

$w=\sin x$

$\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dw}.\frac{dw}{dx}$

$=2u.\sec^2(v).3w^2.\cos x$

$=2\tan(\sin^3 x).\sec^2(\sin^3 x).3\sin^2 x.\cos x$ //

i') $y = \sec\sqrt{x} \cdot \tan\frac{1}{x}$

$y' = (\sec\sqrt{x})' \cdot \tan\frac{1}{x} + \sec\sqrt{x} \cdot \left(\tan\frac{1}{x}\right)'$

$= \sec\sqrt{x} \cdot \tan\sqrt{x} \cdot \frac{1}{2\sqrt{x}} \cdot \tan\frac{1}{x} + \sec\sqrt{x} \cdot \sec^2\frac{1}{x} \cdot \frac{-1}{x^2}$

6) $y = \cot\left(\frac{\sin x}{x}\right)$

quotient rule

$y' = -\csc^2\left(\frac{\sin x}{x}\right) \cdot \left(\frac{\sin x}{x}\right)' = -\csc^2\left(\frac{\sin x}{x}\right) \cdot \frac{(\cos x)\, x - \sin x}{x^2}$

b) $y = \left(\frac{\sin x}{1+\cos x}\right)^2$

$y' = 2 \cdot \left(\frac{\sin x}{1+\cos x}\right) \cdot \left(\frac{\sin x}{1+\cos x}\right)'$

$= 2 \cdot \left(\frac{\sin x}{1+\cos x}\right) \cdot \frac{\cos x\,(1+\cos x) - \sin x\,(-\sin x)}{(1+\cos x)^2} = \frac{2\sin x}{(1+\cos x)^2}$

c) $y = x^{-3}\sec^2(2x)$

$y' = -3 \cdot (x^{-4}) \cdot \sec^2(2x) + x^{-3} \cdot 2\sec(2x) \cdot \underbrace{(\sec 2x)(\tan 2x) \cdot 2}_{(\sec 2x)'}$

d) $y = \frac{\tan x}{1+\tan x} \Rightarrow y' = \frac{\sec^2 x\,(1+\tan x) - (\tan x)(\sec^2 x)}{(1+\tan x)^2} = \frac{\sec^2 x}{(1+\tan x)^2}$

e) $y = \left(-1 - \frac{\csc\theta}{2} - \frac{\theta^2}{4}\right)^2$

$y' = 2\left(-1 - \frac{\csc\theta}{2} - \frac{\theta^2}{4}\right)\left(\frac{\csc\theta\cot\theta}{2} - \frac{\theta}{2}\right)$

f) $y' = 4(1-x)^3(-1)(1+\sin^2 x)^{-5} + (1-x)^4 \cdot -5(1+\sin^2 x)^{-6} \cdot 2\sin x \cdot \cos x$

7) $y' = 6x^2 - 6x - 12$

a) perpendicular to the line $y = 1 - \frac{x}{24}$, slope $= -\frac{1}{24}$

the slope of tangent line $m_T = 24$ since orthogonality

$6x^2 - 6x - 12 = 24 \Rightarrow x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$ $x = 3$ or $x = -2$

b) parallel to the line $y = \sqrt{2} - 12x$, slope $= -12$

the slope of tangent line $m_T = -12$

$6x^2 - 6x - 12 = -12 \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0$ $x = 0$,

or $x = 1$

Q: Use implicit differentiation to find $\frac{dy}{dx}$.

a) $2xy + y^2 = x + y$ b) $y^2 \cdot \cos\left(\frac{1}{y}\right) = 2x + 2y$

<u>Solution</u> a) $2xy + y^2 = x + y$, We cannot solve the equation for y as an explicit function of x,

So, we must use implicit differentiation.

$$\frac{d}{dx}(2xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(x) + \frac{d}{dx}(y)$$

$$2 \cdot 1 \cdot y + 2x \cdot \frac{dy}{dx} + 2 \cdot y \cdot \frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$2y + 2x \cdot y' + 2y \cdot y' = 1 + y'$$

$$y'(2x + 2y - 1) = 1 - 2y$$

$$y' = \frac{1 - 2y}{2x + 2y - 1} \quad //$$

b) $$\frac{d}{dx}\left(y^2 \cdot \cos\left(\frac{1}{y}\right)\right) = \frac{d}{dx}(2x + 2y)$$

$$\frac{d}{dx}(y^2) \cdot \cos\left(\frac{1}{y}\right) + y^2 \cdot \frac{d}{dx}\left(\cos\left(\frac{1}{y}\right)\right) = 2 + 2 \cdot \frac{dy}{dx}$$

$$2 \cdot y \cdot \frac{dy}{dx} \cdot \cos\left(\frac{1}{y}\right) + y^2 \cdot \left[-\sin\left(\frac{1}{y}\right) \cdot \frac{-1}{y^2} \cdot \frac{dy}{dx}\right] = 2 + 2 \cdot \frac{dy}{dx}$$

$$2y \cdot y' \cdot \cos\left(\frac{1}{y}\right) + y^2 \cdot \sin\left(\frac{1}{y}\right) \cdot \frac{1}{y^2} \cdot y' - 2y' = 2$$

$$y' = \frac{2}{2y \cdot \cos\left(\frac{1}{y}\right) + y^2 \cdot \sin\left(\frac{1}{y}\right) \cdot \frac{1}{y^2} - 2}$$

Q: Find the value of $\frac{d^2y}{dx^2}$ for the following function

$xy + y^2 = 1$ at the point $(0,-1)$.

<u>Solution:</u>

$$\frac{d}{dx}(xy + y^2) = \frac{d}{dx}(1)$$

$$1.y + x.y' + 2yy' = 0 \Rightarrow y' = \frac{-y}{x+2y}$$

Take second derivative,

$$y' + 1.y' + x.y'' + 2y'.y' + 2y.y'' = 0$$

$$y''(x+2y) = -2y' - 2(y')^2$$

$$y'' = \frac{-2y' - 2(y')^2}{x+2y}$$

$$y'' = \frac{-2\left(\frac{-y}{x+2y}\right) - 2\left(\frac{-y}{x+2y}\right)^2}{x+2y}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(x,y)=(0,-1)} = \frac{-2\left(\frac{1}{-2}\right) - 2\left(\frac{1}{-2}\right)^2}{-2}$$

$$= \frac{+1 - \frac{1}{2}}{-2} = -\frac{1}{4} \,//$$

Q: Find all the points on the curve which have ~~m~~ slope -1

$$x^2y^2 + xy = 2$$

<u>Solution</u>: We need to find the points (x_0, y_0) such that $y'\Big|_{(x_0,y_0)} = -1$

To find y', we'll use implicit differentation, because we cannot solve the equation for y as an explicit function of x.

$$\frac{d}{dx}(x^2y^2) + \frac{d}{dx}(xy) = \frac{d}{dx}(2)$$

$$2x.y^2 + x^2.2.y.y' + 1.y + xy' = 0$$

$$2x_0.y_0^2 + x_0^2\,2y_0.(-1) + y_0 + x_0(-1) = 0$$

$$2x_0y_0^2 - 2x_0^2y_0 + y_0 - x_0 = 0.$$

$$2x_0y_0(y_0 - x_0) + (y_0 - x_0) = 0.$$

$$(2x_0y_0 + 1)(y_0 - x_0) = 0 \Rightarrow y_0 = x_0 \quad \text{or} \quad x_0y_0 = -\frac{1}{2}$$

- For $x_0 = y_0$, $\quad x_0^2x_0^2 + x_0x_0 = 2$

$$x_0^4 + x_0^2 - 2 = 0$$

$$(x_0^2+2)(x_0^2-1) = 0 \Rightarrow x_0^2 = 1 \quad x_0 = 1, \; x_0 = -1$$

$$x_0^2 \neq -2$$

Thus, $(1,1)$ or $(-1,-1)$

- For $x_0y_0 = -\frac{1}{2}$, $\quad \left(-\frac{1}{2}\right)^2 - \frac{1}{2} \neq 2$

Q: Find an equation for the tangent line to each of the following parametrized curves at the given value. Also find the value of $\frac{d^2y}{dx^2}$ at the given point.

a) $x = \sec^2 t - 1$, $y = \tan t$, $t = -\frac{\pi}{4}$

b) $x = -\sqrt{t+1}$, $y = \sqrt{3t}$, $t = 3$.

Solution: a) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sec^2 t}{2\sec t . \sec t . \tan t} = \frac{1}{2\tan t}$

$$\frac{dy}{dx}\bigg|_{t=-\frac{\pi}{4}} = \frac{1}{2\tan(-\frac{\pi}{4})} = -\frac{1}{2}\,{}_{\prime\prime} \quad \text{(slope of tangent line)}$$

if $t = -\frac{\pi}{4}$, then $x = \sec^2(-\frac{\pi}{4}) - 1 = \frac{1}{\cos^2(-\frac{\pi}{4})} - 1 = 1$

$$y = \tan(-\tfrac{\pi}{4}) = -1$$

$m = -\frac{1}{2}$ $(x_0, y_0) = (1, -1)$

The equation of tangent line is that $y = -1 + (-\frac{1}{2})(x-1)$

$$2y = -1 - x \,{}_{\prime\prime}$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\left[(2\tan t)^{-1}\right]'}{2\sec^2 t . \tan t} = \frac{-1(2\tan t)^{-2} . 2\sec^2 t}{2\sec^2 t . \tan t}$$

$$= -\frac{1}{4\tan^3 t}$$

$$\frac{d^2y}{dx^2}\bigg|_{t=-\frac{\pi}{4}} = \frac{1}{4}\,{}_{\prime\prime}$$

b) $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{1}{2\sqrt{3t}}\cdot 3}{-\frac{1}{2\sqrt{t+1}}} = \frac{-3\sqrt{t+1}}{\sqrt{3t}} = -3\sqrt{\frac{t+1}{3t}}$$

$$\left.\frac{dy}{dx}\right|_{t=3} = \frac{-3\cdot 2}{3} = -2 \,//$$

when $t=3$, $x = -\sqrt{3+1} = -2$ $\quad (-2, 3)$

$y = \sqrt{3\cdot 3} = 3$

The equ. of tangent line : $y = 3 + (-2)(x+2)$

$y = -1 - 2x \,//$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\left[-3\cdot\left(\frac{t+1}{3t}\right)^{1/2}\right]'}{-\frac{1}{2\sqrt{t+1}}} = 6\sqrt{t+1}\cdot\left[\left(\frac{t+1}{3t}\right)^{1/2}\right]'$$

$$= 6\sqrt{t+1}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{\frac{t+1}{3t}}}\cdot\frac{3t-(t+1)3}{9t^2}$$

$$= 6\cdot\frac{1}{2}\cdot 3\cdot\frac{1}{9}\cdot\sqrt{t+1}\cdot\frac{\sqrt{3t}}{\sqrt{t+1}}\cdot\frac{-1}{t^2}$$

$$= -\frac{\sqrt{3t}}{t^2}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=3} = -\frac{3}{9} = -1/3 \,//$$

Q: Assuming that the following equations define x and y implicitly as differentiable functions $x=f(t)$ and $y=g(t)$.

$$x\sin t+\sqrt{x}=t\ ,\quad t\sin t-2t=y\ ,\quad t=\pi$$

Find the slope of the curve $x=f(t)$, $y=g(t)$ at the given value of t, and write the tangent line at $t=\pi$

Solution: We'll calculate the value of $\left.\frac{dy}{dx}\right|_{t=\pi}$

Slope: $$\left.\frac{dy}{dx}\right|_{t=\pi}=\left.\frac{dy/dt}{dx/dt}\right|_{t=\pi}$$

$$\left.\frac{dy}{dt}\right|_{t=\pi}=\left.(\sin t+t\cos t-2)\right|_{t=\pi}=\sin\pi+\pi.\cos\pi-2=-\pi-2$$

$$\left.\frac{dx}{dt}\right|_{t=\pi}=\ ?\qquad \frac{d}{dt}(x.\sin t)+\frac{d}{dt}(\sqrt{x})=\frac{d}{dt}(t)$$

$$\frac{dx}{dt}.\sin t+x.\underbrace{\frac{d}{dt}(\sin t)}_{\cos t}+\frac{1}{2\sqrt{x}}.\frac{dx}{dt}=1$$

$$\frac{dx}{dt}=\frac{1-x.\cos t}{\sin t+\frac{1}{2\sqrt{x}}}$$

when $t=\pi$, $\quad x.\sin\pi+\sqrt{x}=\pi \Rightarrow x=\pi^2$

$$\pi.\sin\pi-2\pi=y \Rightarrow y=-2\pi$$

$$\left.\frac{dx}{dt}\right|_{t=\pi}=\frac{1-\pi^2.\cos\pi}{\sin\pi+\frac{1}{2\sqrt{\pi^2}}}=\frac{1-\pi^2}{\frac{1}{2\pi}}=2\pi-2\pi^3$$

$$\left.\frac{dy}{dx}\right|_{t=\pi}=\frac{-\pi-2}{2\pi-2\pi^3}$$

Tangent line:

$$y=-2\pi+\left(\frac{-\pi-2}{2\pi-2\pi^3}\right)(x-\pi^2)$$ //

5

Q: Find the $\frac{dy}{dx}$ for the following functions

chain rule
outside-inside rule
$y = f(g(x))$
$y' = f'(g(x)).g'(x)$

a) $y = \tan(x + \cos x)$

b) $y = \tan^2(\sin^3 x)$

c) $y = \cos\left(5 \sin \frac{x}{3}\right)$

Solution

a) $y = \tan u$

$u = x + \cos x$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \sec^2 u \cdot (1 - \sin x)$$

$$= \sec^2(x + \cos x) \cdot (1 - \sin x)$$

b) $y = u^2$

$u = \tan v$

$v = t^3$

$t = \sin x$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dt} \cdot \frac{dt}{dx}$$

$$= 2u . \sec^2 v . 3t^2 . \cos x$$

$$= 2 . \tan(\sin^3 x) . \sec^2(\sin^3 x) . 3 . \sin^2 x . \cos x$$

$$y' = 2 \tan(\sin^3 x) . [\tan(\sin^3 x)]'$$

$$= 2 \tan(\sin^3 x) . \sec^2(\sin^3 x) . (\sin^3 x)'$$

$$= 2 \tan(\sin^3 x) . \sec^2(\sin^3 x) . 3 \sin^2 x . \cos x$$

c) $y = \cos u$

$u = 5 \sin v$

$v = \frac{x}{3}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$= -\sin u . 5 \cos v . \frac{1}{3}$$

$$= -\sin\left(5 \sin \frac{x}{3}\right) . 5 \cos \frac{x}{3} . \frac{1}{3}$$

Q: Find $\frac{d^2y}{dx^2}$ for the following implicit function $y^2 = 1 - \frac{2}{x}$

~~b) evaluate $\frac{d^2y}{dx^2}$ at the point (~~

Solution: $\frac{d}{dx}(y^2) = \frac{d}{dx}(1 - 2.x^{-1})$

$$2.y.y' = (-2)(-1)\,x^{-2}$$

$$2y.y' = 2x^{-2} \Rightarrow y' = \frac{1}{x^2.y}$$

Take second derivative.

$$\frac{d}{dx}(2y.y') = \frac{d}{dx}(2x^{-2})$$

$$\frac{d}{dx}(2y).y' + 2y.\frac{d}{dx}(y') = 2(-2)\,x^{-3}$$

$$2y'.y' + 2y.y'' = -4x^{-3}$$

$$y'' = \frac{-4x^{-3} - 2(y')^2}{2y}$$

$$y'' = \frac{-4x^{-3} - 2(x^{-2}.y^{-1})^2}{2y} = \frac{-4x^{-3} - 2x^{-4}y^{-2}}{2y}$$

$$= -\frac{2xy^2 + 1}{x^4y^3} \quad //$$