Homework 3 MIA503E Due: April 17, 2018

1) Blasius solution to the laminar boundary layer equations yields

$$v=\frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}(\eta f'-f)$$

For the vertical velocity where the stream function $\psi=f(\eta)\sqrt{\nu x U_\infty}$ and $\eta=y\sqrt{\frac{U_\infty}{x\nu}}$

a) Verify the expression for v

b) Find the x component of the acceleration a_x in terms of $f, x\ and\ U_\infty$

c) Find the expression for the wall shear stress in terms of f and η

d) Find an algebraic expression for the total viscous drag for a flat plate of length L and width w

Solution:

a)

$$\frac{\partial\eta}{\partial x}=-\frac{y}{2}\sqrt{\frac{U_\infty}{x^3\nu}},\frac{\partial\eta}{\partial y}=\sqrt{\frac{U_\infty}{x\nu}},u=\frac{\partial\psi}{\partial y}=f'(\eta)\frac{\partial\eta}{\partial y}\sqrt{\nu x U_\infty}=f'(\eta)\sqrt{\frac{U_\infty}{x\nu}}\sqrt{\nu x U_\infty}=U_\infty f'(\eta)$$

$$v=-\frac{\partial\psi}{\partial x}=-\left(f'(\eta)\frac{\partial\eta}{\partial x}\sqrt{\nu x U_\infty}+\frac{f(\eta)}{2}\sqrt{\frac{\nu U_\infty}{x}}\right)=-\left(f'(\eta)\left(-\frac{y}{2}\sqrt{\frac{U_\infty}{x^3\nu}}\right)\sqrt{\nu x U_\infty}+\frac{f(\eta)}{2}\sqrt{\frac{\nu U_\infty}{x}}\right)$$

$$=\frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\left(f'(\eta)\left(y\sqrt{\frac{U_\infty}{x\nu}}\right)-f(\eta)\right)=\frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}(\eta f'-f)$$

b)

$$a_x=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=0+U_\infty f'(\eta)U_\infty f''(\eta)\frac{\partial\eta}{\partial x}+\frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}(\eta f'-f)U_\infty f''(\eta)\frac{\partial\eta}{\partial y}$$

$$=U_\infty f'(\eta)U_\infty f''(\eta)\left(-\frac{y}{2}\sqrt{\frac{U_\infty}{x^3\nu}}\right)+\frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}(\eta f'-f)U_\infty f''(\eta)\sqrt{\frac{U_\infty}{x\nu}}$$

$$=\frac{{U_\infty}^2 f''(\eta)}{2x}\left(f'(\eta)\left(-y\sqrt{\frac{U_\infty}{x\nu}}\right)+(\eta f'-f)\right)=-\frac{{U_\infty}^2 ff''}{2x}$$

c)

$$\tau=\mu\frac{\partial u}{\partial y}=\mu U_\infty f''(\eta)\frac{\partial\eta}{\partial y}=\mu U_\infty f''(\eta)\sqrt{\frac{U_\infty}{x\nu}},\quad \tau_w=\frac{\mu {U_\infty}^{\frac{3}{2}}f''(0)}{\sqrt{x\nu}}$$

d) Drag on one side

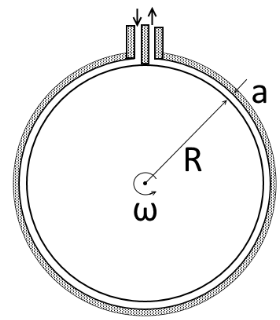
$$\boldsymbol{F}=w\int_0^L\tau_w dx=w\frac{\mu {U_\infty}^{\frac{3}{2}}f''(0)}{\sqrt{\nu}}\int_0^L\frac{dx}{\sqrt{x}}=w\frac{\mu {U_\infty}^{\frac{3}{2}}f''(0)}{\sqrt{\nu}}2\sqrt{L}=2f''(0)w{U_\infty}^{\frac{3}{2}}\sqrt{\rho\mu L}=\frac{2f''(0)}{\sqrt{Re}}\rho {U_\infty}^2 wL$$

2) Consider a viscous shear pump made from a stationary housing with a close-fitting rotating drum inside. The clearance $\boldsymbol{a}$ is small compared to the radius $\boldsymbol{R}$ so that flow in the annular space may be treated as flow between parallel plates.

a) Find the pressure differential Δp

b) Input power P_{in}

c) Power output P_{out} and efficiency P_{out}/P_{in}
As functions of volumetric flow rate per unit length
(Q/b where b is the length of the drum)



Solution:

a)

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2} \rightarrow \frac{\partial u}{\partial y} = \frac{1}{\mu}\frac{\partial p}{\partial x} y + c_1 \rightarrow u = \frac{1}{2\mu}\frac{\partial p}{\partial x} y^2 + c_1 y + c_2$$

$$y = 0, u = 0 \rightarrow c_2 = 0, \qquad y = a, u = \omega R \rightarrow \omega R = \frac{1}{2\mu}\frac{\partial p}{\partial x} a^2 + c_1 a \rightarrow c_1 = \frac{\omega R}{a} - \frac{1}{2\mu}\frac{\partial p}{\partial x} a$$

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x} y^2 + \left(\frac{\omega R}{a} - \frac{1}{2\mu}\frac{\partial p}{\partial x} a\right) y = \frac{1}{2\mu}\frac{\partial p}{\partial x}(y^2 - ay) + \frac{\omega R}{a} y$$

$$\frac{Q}{b} = \int_0^a u dy = \left(\frac{1}{2\mu}\frac{\partial p}{\partial x}\left(\frac{y^3}{3} - a\frac{y^2}{2}\right) + \frac{\omega R}{a}\frac{y^2}{2}\right)\Big|_0^a = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(-\frac{a^3}{6}\right) + \frac{\omega R}{a}\frac{a^2}{2}$$

$$\frac{1}{2\mu}\frac{\partial p}{\partial x}\left(-\frac{a^3}{6}\right) = \frac{Q}{b} - \frac{a\omega R}{2}, \qquad -\frac{\partial p}{\partial x} = -\frac{\Delta p}{2\pi R} = \frac{12\mu}{a^3}\left(\frac{Q}{b} - \frac{a\omega R}{2}\right)$$

$$\Delta p = \frac{24\pi R\mu}{a^3}\left(\frac{a\omega R}{2} - \frac{Q}{b}\right) = \frac{12\pi R^2 \mu\omega}{a^2}\left(1 - \frac{2Q}{a\omega R b}\right)$$

b) Per unit length power

$$P_{in} = \frac{T\omega}{b} = \tau_w L R\omega = \mu \frac{\partial u}{\partial y}\Big|_{y=a} L R\omega = \mu\left(\frac{\omega R}{a} + \frac{1}{2\mu}\frac{\Delta p}{2\pi R} a\right) 2\pi R^2 \omega$$

$$P_{in} = \mu\left(\frac{2\pi\omega R^2}{a} + \frac{a\Delta p}{2\mu}\right) R\omega = \mu\left(\frac{2\pi\omega R^2}{a} + \frac{6\pi R^2\omega}{a}\left(1 - \frac{2Q}{a\omega R b}\right)\right) R\omega$$

c)

$$P_{out} = \Delta p \frac{Q}{b} = \frac{12\pi R^2 \mu\omega}{a^2}\left(1 - \frac{2Q}{a\omega R b}\right)\frac{Q}{b}$$

$$\eta = \frac{P_{out}}{P_{in}} = \frac{\frac{6}{a}\left(1 - \frac{2Q}{a\omega R b}\right)\frac{Q}{b}}{\left(1 + 3\left(1 - \frac{2Q}{a\omega R b}\right)\right) R\omega} = \frac{\left(1 - \frac{2Q}{a\omega R b}\right)}{\left(4 - \frac{6Q}{a\omega R b}\right)}\frac{6Q}{a\omega R b}$$

$$\eta_{max} = \frac{1}{3} \; at \; \frac{Q}{a\omega R b} = \frac{1}{3}$$

3) A spherical particle, under the influence of gravity, falls very slowly through a viscous fluid.
   a) Find the terminal velocity of the particle in terms of $\rho_{\text{particle}}, \rho_{\text{fluid}}, D \ and \ \mu$.
   b) Calculate the velocity for the given values below and check the validity of your assumptions

$\rho_{\text{particle}} = 4000 \frac{\text{kg}}{\text{m}^3}, \quad \rho_{\text{fluid}} = 800 \frac{\text{kg}}{\text{m}^3}, \ D = 0.5\, mm, \quad \mu = 0.1 \frac{kg}{m.s}$

Solution:

$$W = (\rho_p - \rho_f)gVol = (\rho_p - \rho_f)g\frac{4}{3}\pi(\frac{D}{2})^3$$

$$D = C_D\frac{1}{2}\rho_f V^2 A = \frac{24}{Re}\frac{1}{2}\rho_f V^2 \pi(\frac{D}{2})^2 = 3\pi\mu V D$$

$$W = D \rightarrow \ V = \frac{g}{18\mu}(\rho_p - \rho_f)D^2 = 0.00436\frac{m}{s}$$

$$Re = \frac{\rho_f V D}{\mu} = 0.0174 \ll 1$$