Homework 2 **MIA503E** **Due: March 27, 2018**

1) Use the momentum equation to derive an equation for the transport of vorticity:

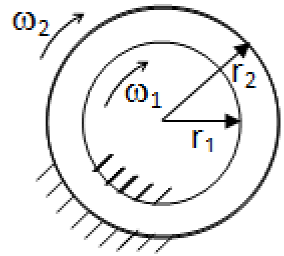
$$\boldsymbol{w_i = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j}}$$

Momentum equation for incompressible flow

$$\boldsymbol{\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j}}$$

Write the result also in vector notation. Which term will disappear in 2D?

2) For the flow between concentric rotating cylinders, find the temperature distribution with the following boundary conditions. $\boldsymbol{\omega_2 = 0}$, $\boldsymbol{T_2}$, $\boldsymbol{\omega_2}$ is given and inner cylinder is adiabatic.



3) The figure for this problem shows a conduit whose cross section is the shape of an equilateral triangle. For the coordinate system shown in the figure, the equations of the three sides are

$$z + \frac{b}{2\sqrt{3}} = 0$$

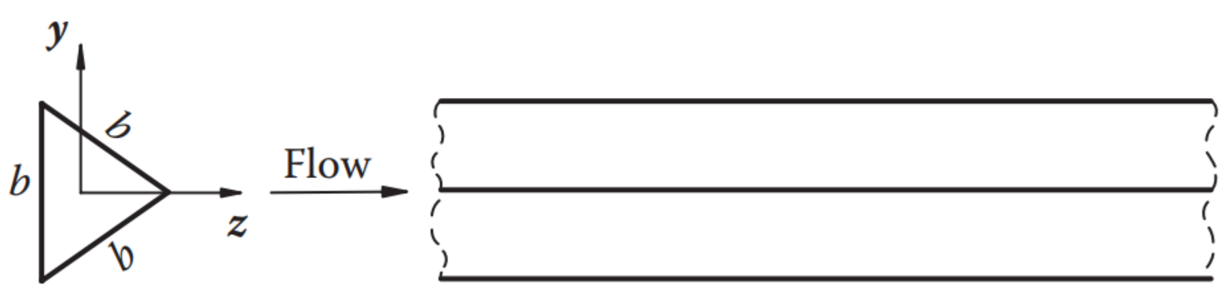
$$z + \sqrt{3}y - \frac{b}{\sqrt{3}} = 0$$

$$z - \sqrt{3}y - \frac{b}{\sqrt{3}} = 0$$

Look for a solution for the velocity distribution in this conduit of the following form:

$$u(y,z) = \alpha(z + \frac{b}{2\sqrt{3}})(z + \sqrt{3}y - \frac{b}{\sqrt{3}})(z - \sqrt{3}y - \frac{b}{\sqrt{3}})$$

Determine the value of the constant α such that the assumed form of solution is exact, with the value of this constant being expressed in terms of the applied pressure gradient.



Solution 1: Panton 4th

$$\partial_0 v_i + v_j \partial_j v_i = -\frac{1}{\rho}\partial_i p + \nu\, \partial_j \partial_j v_i \tag{13.3.1}$$

Into this equation we substitute the vector identity (Problem 3.15)

$$v_j \partial_j v_i = \partial_i \left(\tfrac{1}{2} v_j v_j\right) + \varepsilon_{ijk}\omega_j v_k \tag{13.3.2}$$

The resulting equation is differentiated with ∂_q and multiplied by ε_{pqj} to yield

$$\begin{aligned}\partial_0(\varepsilon_{pqi}\partial_q v_i) + \varepsilon_{pqi}\partial_q \partial_i (\tfrac{1}{2} v_j v_j) + \varepsilon_{pqi}\partial_q (\varepsilon_{ijk}\omega_j v_k) \\ = -\frac{1}{\rho}\varepsilon_{pqi}\partial_q \partial_i p + \nu \varepsilon_{pqi}\partial_j \partial_j \partial_q v_i\end{aligned} \tag{13.3.3}$$

Consider this equation term by term. The first term can be identified as the time derivative of the vorticity. The second term is zero because antisymmetric ε_{pqi} is multiplied by symmetric $\partial_q \partial_i$. For the same reason the pressure term on the right-hand side is zero. Also note that the last term contains the vorticity. The term we skipped is expanded to yield (the last line below is obtained by noting that $\partial_k v_k$ and $\partial_j \omega_j$ are always zero)

$$\begin{aligned}\varepsilon_{pqi}\varepsilon_{ijk}\partial_q (\omega_j v_k) &= \partial_k(\omega_p v_k) - \partial_j(\omega_j v_p) \\ &= v_k \partial_k \omega_p - \omega_j \partial_j v_p\end{aligned} \tag{13.3.4}$$

Collecting these results yields the final vorticity transport equation:

$$\partial_0 \omega_i + v_j \partial_j \omega_i = \omega_j \partial_j v_i + \nu \partial_j \partial_j \omega_i$$

or in symbolic notation,

$$\underbrace{\frac{D\boldsymbol{\omega}}{Dt}}_{\text{rate of change of particle vorticity}} = \underbrace{\boldsymbol{\omega}\cdot\nabla\mathbf{v}}_{\text{rate of deforming vortex lines}} + \underbrace{\nu\,\nabla^2\boldsymbol{\omega}}_{\text{net rate of viscous diffusion of }\boldsymbol{\omega}} \tag{13.3.5}$$

In 2D $\vec{\omega}\cdot\vec{\nabla}\vec{V}=0$

Solution 2:

$$T(r)=\frac{\mu\, r_1{}^4 r_2{}^2 w1^2 (r-r_2)(r+r_2)}{k r^2 (r_1{}^2-r_2{}^2)^2}+\frac{2\mu\, r_1{}^2 r_2{}^4 w1^2 (log(r_2)-log(r))}{k(r_1{}^2-r_2{}^2)^2}+T_2$$

$$T(r)=\frac{\mu\, r_1{}^2 r_2{}^2 w1^2}{k(r_1{}^2-r_2{}^2)^2}\left[r_1{}^2\left(1-\left(\frac{r_2}{r}\right)^2\right)+2\,r_2{}^2\log\left(\frac{r_2}{r}\right)\right]+T_2$$

$$T(r)=\frac{\mu\, r_1{}^4 r_2{}^4 w1^2}{k(r_1{}^2-r_2{}^2)^2}\left[\left(\frac{1}{r_2{}^2}-\frac{1}{r^2}\right)+\frac{2}{r_1{}^2}\log\left(\frac{r_2}{r}\right)\right]+T_2$$

Solution 3:

$$\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{\mu}\frac{\partial p}{\partial x}$$

$$u(y,z)=\alpha(z+\frac{b}{2\sqrt{3}})(z+\sqrt{3}y-\frac{b}{\sqrt{3}})(z-\sqrt{3}y-\frac{b}{\sqrt{3}})$$

$$2\alpha\left(-\frac{b}{\sqrt{3}}+\sqrt{3}y+z\right)+2\alpha\left(\frac{b}{2\sqrt{3}}-\sqrt{3}y+2z\right)-6\alpha\left(\frac{b}{2\sqrt{3}}+z\right)=\frac{1}{\mu}\frac{\partial p}{\partial x}$$

$$\alpha=-\frac{1}{2\sqrt{3}b\mu}\frac{\partial p}{\partial x}$$