Homework 1 Solution MIA503E Due: March 14, 2018

1) Show that divergence of vorticity is zero.
   Vorticity w_i and its divergence is given as follows in tensor notation

$$w_i = \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}, \qquad \frac{\partial w_i}{\partial x_i} = 0$$

Solution:

$$\frac{\partial}{\partial x_i}\varepsilon_{ijk}\frac{\partial u_k}{\partial x_j} = \frac{\partial}{\partial x_1}\varepsilon_{123}\frac{\partial u_3}{\partial x_2} + \frac{\partial}{\partial x_1}\varepsilon_{132}\frac{\partial u_2}{\partial x_3}$$
$$+\frac{\partial}{\partial x_2}\varepsilon_{231}\frac{\partial u_1}{\partial x_3} + \frac{\partial}{\partial x_2}\varepsilon_{213}\frac{\partial u_3}{\partial x_1}$$
$$+\frac{\partial}{\partial x_3}\varepsilon_{312}\frac{\partial u_2}{\partial x_1} + \frac{\partial}{\partial x_3}\varepsilon_{321}\frac{\partial u_1}{\partial x_2}$$
$$= \frac{\partial}{\partial x_1}\frac{\partial u_3}{\partial x_2} - \frac{\partial}{\partial x_1}\frac{\partial u_2}{\partial x_3} + \frac{\partial}{\partial x_2}\frac{\partial u_1}{\partial x_3} - \frac{\partial}{\partial x_2}\frac{\partial u_3}{\partial x_1} + \frac{\partial}{\partial x_3}\frac{\partial u_2}{\partial x_1} - \frac{\partial}{\partial x_3}\frac{\partial u_1}{\partial x_2} = 0$$

2) Write the following identity in tensor notation and prove it

$$(\vec{V}\cdot\vec{\nabla})\vec{V} = \nabla\left(\frac{V^2}{2}\right) - \vec{V}\times\vec{\nabla}\times\vec{V}$$

Hint: use the identity $\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$

Solution:

$$u_j\frac{\partial u_i}{\partial x_j} = \frac{\partial(\frac{u_j u_j}{2})}{\partial x_i} - \varepsilon_{ijk}u_j\varepsilon_{klm}\frac{\partial u_m}{\partial x_l}$$

$$u_j\frac{\partial u_i}{\partial x_j} = u_j\frac{\partial u_j}{\partial x_i} - (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})u_j\frac{\partial u_m}{\partial x_l}$$

$$u_j\frac{\partial u_i}{\partial x_j} = u_j\frac{\partial u_j}{\partial x_i} - \delta_{il}\frac{\partial u_m}{\partial x_l}u_j\delta_{jm} + \delta_{im}\frac{\partial u_m}{\partial x_l}u_j\delta_{jl}$$

$$u_j\frac{\partial u_i}{\partial x_j} = u_j\frac{\partial u_j}{\partial x_i} - \frac{\partial u_m}{\partial x_i}u_m + \frac{\partial u_i}{\partial x_l}u_l$$

$$u_j\frac{\partial u_i}{\partial x_j} = u_j\frac{\partial u_j}{\partial x_i} - \frac{\partial u_j}{\partial x_i}u_j + \frac{\partial u_i}{\partial x_j}u_j = \frac{\partial u_i}{\partial x_j}u_j$$

3) Use the following form of the momentum equation to derive an equation for kinetic energy per volume: $\boldsymbol{\kappa = \rho u_i u_i / 2}$

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ji}}{\partial x_j}$$

Result should be in the following form

$$\frac{\partial \kappa}{\partial t} + \frac{\partial (u_i \kappa)}{\partial x_i} = ?$$

Solution:

$$\rho u_i \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -u_i \frac{\partial p}{\partial x_i} + u_i \frac{\partial \tau_{ji}}{\partial x_j}$$

$$\rho u_i \frac{\partial u_i}{\partial t} + \rho u_j u_i \frac{\partial u_i}{\partial x_j} = -u_i \frac{\partial p}{\partial x_i} + u_i \frac{\partial \tau_{ji}}{\partial x_j}$$

$$\rho \frac{\partial}{\partial t}\left(\frac{u_i u_i}{2}\right) + \rho u_j \frac{\partial}{\partial x_j}\left(\frac{u_i u_i}{2}\right) = -u_i \frac{\partial p}{\partial x_i} + u_i \frac{\partial \tau_{ji}}{\partial x_j}$$

$$\frac{\partial}{\partial t}\left(\rho \frac{u_i u_i}{2}\right) - \frac{u_i u_i}{2}\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho u_j \frac{u_i u_i}{2}\right) - \frac{u_i u_i}{2}\frac{\partial}{\partial x_j}(\rho u_j) = -u_i \frac{\partial p}{\partial x_i} + u_i \frac{\partial \tau_{ji}}{\partial x_j}$$

$$\frac{\partial}{\partial t}\left(\rho \frac{u_i u_i}{2}\right) + \frac{\partial}{\partial x_j}\left(\rho u_j \frac{u_i u_i}{2}\right) - \frac{u_i u_i}{2}\left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j}\right) = -u_i \frac{\partial p}{\partial x_i} + u_i \frac{\partial \tau_{ji}}{\partial x_j}$$

$$\frac{\partial \kappa}{\partial t} + \frac{\partial (u_i \kappa)}{\partial x_i} = = -u_i \frac{\partial p}{\partial x_i} + u_i \frac{\partial \tau_{ji}}{\partial x_j}$$