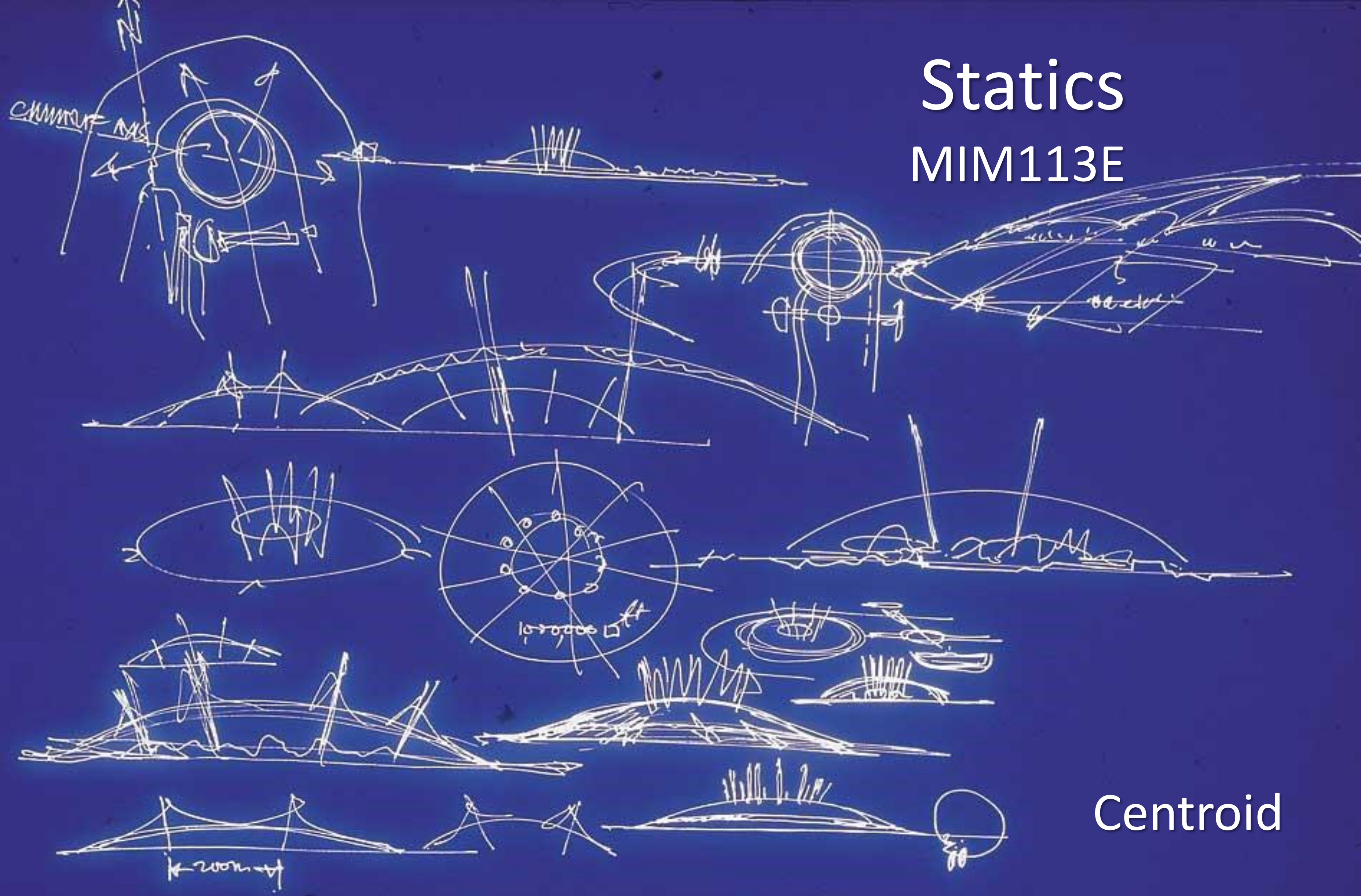

Statics

MIM113E

Centroid

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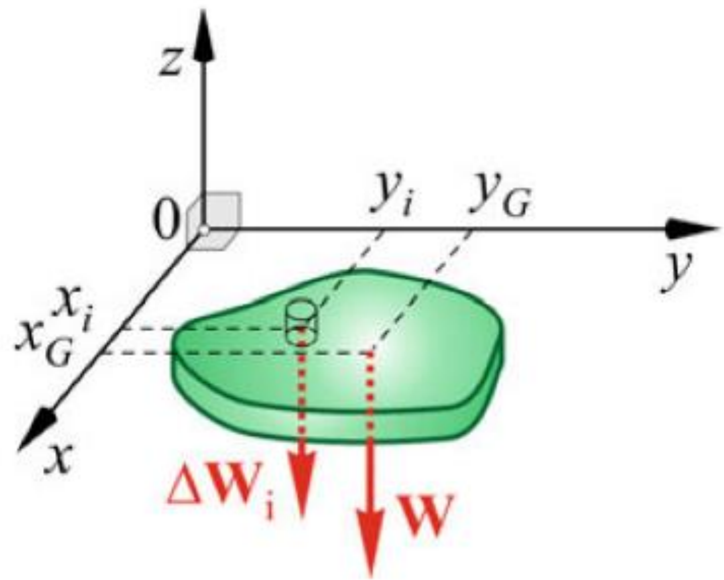
Fig. 6.2 Locating of the center of gravity for a two-dimensional rigid body



each element by its coordinates x_i and y_i, while the gravitational force exerted by the Earth on it is defined as $\Delta \mathbf{W}_i$, where i is the number of an element. The resultant force (weight) of all these elements is defined by

$$\mathbf{W} = \sum_{i} \Delta \mathbf{W}_{i}$$

The location of the resultant force (line of action) may be determined by summing the moments of each $\Delta\mathbf{W}i$ about both axes and equating them to the moment of the resultant $\mathbf{W}$ about the same axes.

About x-axis

$$\sum_{\mathrm{i}} \Delta\mathbf{M}_{\mathrm{i}} = \sum_{\mathrm{i}} \Delta\mathbf{W}_{\mathrm{i}} \cdot \mathrm{y}_i = \mathbf{W} \cdot y_{\mathrm{G}} \tag{6.3}$$

and about y-axis,

$$\sum_{i} \Delta M_i = \sum_{i} \Delta\mathbf{W}_{\mathrm{i}} \cdot x_i = \mathbf{W} \cdot x_{\mathrm{G}} \tag{6.4}$$

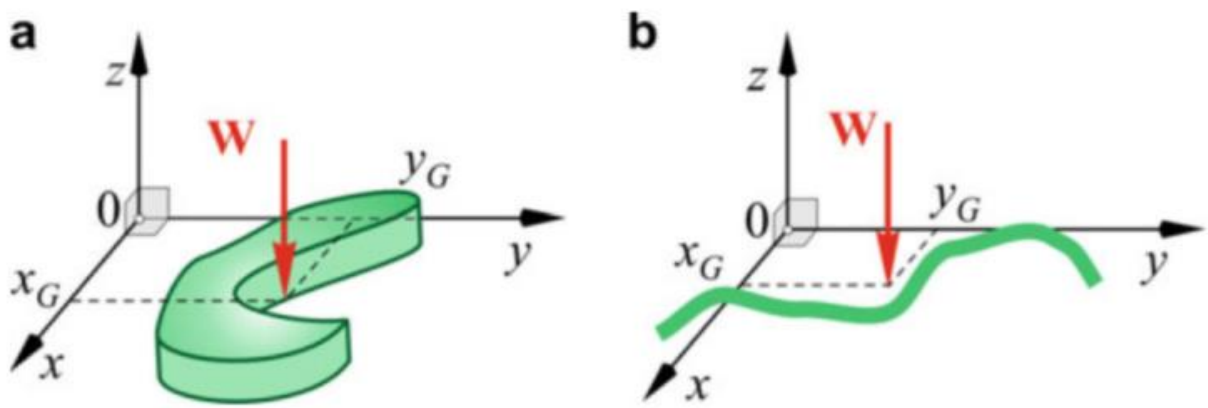


Fig. 6.3 Center of gravity for bodies with different shapes (**a**) and for wires (**b**)

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Here, x_G and y_G are the coordinates of the application of the resultant force **W**. This point defines the center of gravity of the two-dimensional body.

We may increase the number of particles representing our body and at the same time decrease their size to obtain an infinite number of infinitesimally small particles. In this case, the following expressions will describe the weight and location of the center of gravity.

$$\mathbf{W} = \int d\mathbf{W} \tag{6.5}$$

$$\int y\, d\mathbf{W} = \mathbf{W} \cdot y_G \tag{6.6}$$

$$\int x\, d\mathbf{W} = \mathbf{W} \cdot x_G \tag{6.7}$$

6.3 Centroids

Let's assume that a body (Fig. 6.3a) has a constant thickness t and is made of homogeneous material (i.e., its physical properties are *not* the function of the location) with a specific weight $\boldsymbol{\gamma}$ (weight per unit volume). The weight of an element $\Delta \mathbf{W}_i$, which occupies volume $\Delta V_i = t\Delta A_i$ may be expressed as

$$\Delta \mathbf{W}_i = t \cdot \boldsymbol{\gamma} \cdot \Delta A_i \tag{6.8}$$

In the limit, equation (6.8) becomes

$$d\mathbf{W} = t \cdot \boldsymbol{\gamma} \cdot dA \tag{6.9}$$

Substituting expression for $d\mathbf{W}$ into (6.5)–(6.7), one gets

$$\int_{\text{area}} x \cdot dA = A \cdot x_G \tag{6.10}$$

6.2 Center of Gravity of a Flat Plate

Any rigid body may be considered to consist of a number of particles, each having a weight, $d\mathbf{W}$ directed toward the center of the Earth. For bodies, that are significantly smaller than the Earth, gravitational forces acting on each particle can be considered parallel. The resultant of these parallel forces is equal to the weight of the body $\mathbf{W}$,

$$\mathbf{W} = \int_{\text{volume}} d\mathbf{W} \tag{6.1}$$

The moment $\mathbf{M}$ of the resultant and the sum of the moments of all gravitational forces acting on the particles, with respect to the same point, should be equal.

$$\mathbf{M} = \int_{\text{volume}} d\mathbf{M} = \int_{\text{volume}} \mathbf{r} \times d\mathbf{W} = \mathbf{r}_G \times \mathbf{W} \tag{6.2}$$

where $\mathbf{r}_G$ defines the location of the center of gravity.

From the above, the location $\mathbf{r}_G$ of the line of action of the resultant can be determined by calculating the sum of moments for parallel forces.

> The force exerted by the Earth, due to gravitation, on a particle or body is defined as its *weight*.

$$\int_{\text{area}} y \cdot dA = A \cdot y_G \tag{6.11}$$

where A is the area of the top surface of a rigid body. It should be noted that the integration over volume is reduced here to the integration by area, since the thickness of the body is constant. The values of x_G and y_G obtained from (6.10) and (6.11) define the location of the centroid of the area of the top surface of the rigid body we consider.

Similar equations can be derived for determination of the centroid of a flat wire (i.e., all points of the wire belong to the same plane).

$$\int_{\text{contour}} y \cdot dl = L \cdot y_G \tag{6.12}$$

$$\int_{\text{contour}} x \cdot dl = L \cdot x_G \tag{6.13}$$

where dl is the length of the infinitesimal element of the wire and L is the total length of the wire.

It is important to understand the difference between the centroid and center of gravity of a body. The centroid is purely geometrical characteristic of a body. Both locations coincide only when a body has uniform thickness and is homogeneous.

The integrals $\int y\,dA$ and $\int x\,dA$ are called the *first moments* of the area with respect to the x- and y-axis.

$$Q_y = \int_{\text{area}} x \cdot dA$$

and

$$Q_x = \int_{\text{area}} y \cdot dA$$

Thus, one can get the location of the centroid by dividing the *first moment* of the area by the area.

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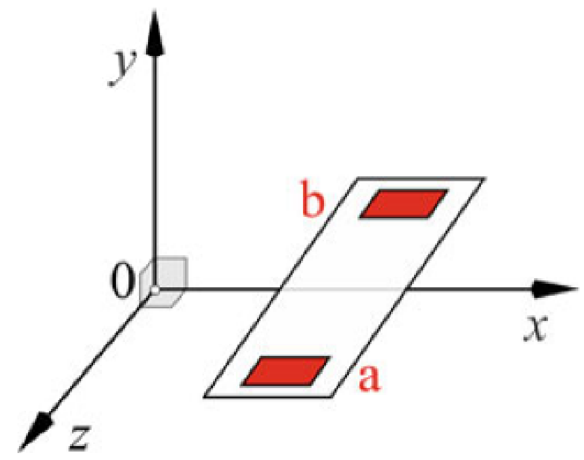
Thus, one can get the location of the centroid by dividing the *first moment* of the area by the area.

$$x_G = \frac{Q_y}{A}$$
$$y_G = \frac{Q_x}{A} \tag{6.14}$$

Fig. 6.4 Axis of symmetry



Area is a scalar quantity and is always positive, while the associated location of the centroid, relative to the selected coordinate system, may be positive or negative, thus the *first moment* of the area may be positive, negative, or zero. If it is zero, it means that the centroid coincides with the coordinate axis. When an area has an axis of symmetry, let say axis x, the centroid is on this axis, since for each element "a" one may find a corresponding element "b" on the other side of the symmetry axis (Fig. 6.4). It is obvious, that if an area has two axes of symmetry the centroid must be on their intersection. This allows us to find centroids of a number of common shapes, having an axis of symmetry, such as circles, squares, and rectangles, without calculations. The same is true for bodies with openings and wires.

Example 6.2 Find the centroid of a rectangle of width b and height h (Fig. 6.6).

Solution Since the rectangle has two axes of symmetry, the centroid should be located at the intersection of its diagonals. However, we will use integration to show this.

Let us choose a vertical strip, as shown, which is located at the distance x from the origin of the coordinate system and has an infinitesimal width of dx. The first moment Q_y of this strip about the y-axis is

$$Q_y = \int_{\text{area}} x dA = \int_0^b x \cdot (h dx) = h \left[\frac{x^2}{2}\right]_0^b = \frac{b^2 h}{2}$$

and $A = b \cdot h$.

From (6.14) we obtain

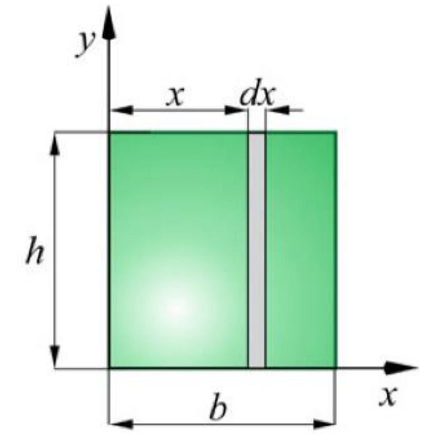


Fig. 6.6 Centroid of a rectangle

$$x_G = \frac{Q_y}{A} = \frac{b}{2}$$

as we have predicted from the symmetry of the rectangle. In the similar manner, we can calculate the value of y_G to be equal to $h/2$.

The first moment for a rectangle about the axis parallel to its base is equal to the base times square of the height divided by two.

Let us call the side of the rectangle along the axis about which we are calculating the 1st moment a base, while the other side we will call a height. Thus, the first moment for a rectangle about the axis parallel to the base is equal to the product of its base by square of height divided by two.

Example 6.3 Find the centroid of a uniform wire in a shape of a quarter of a circle with radius R (Fig. 6.7).

Solution We will use the formulae (6.12) and (6.13). Let us select an infinitesimal element of the length $dl = R \cdot d\theta$ at the location defined by an angle θ, as shown in Fig. 6.7. Centroid of this element is

$$x = R \cdot \cos\theta \quad \text{and} \quad y = R \cdot \sin\theta$$

The total length of the quarter arc is

$$L = \frac{\pi R}{2}$$

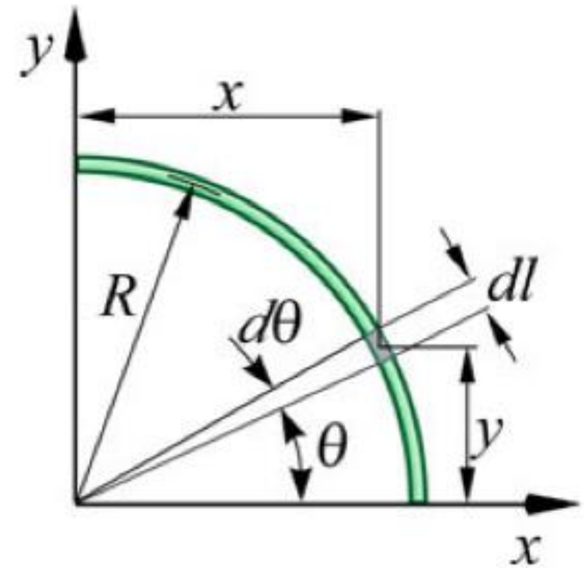


Fig. 6.7 Centroid of a wire

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Thus, the centroid can be calculated from the following expression:

$$x_G = \frac{\int_0^L x \cdot dl}{L} = \frac{\int_0^{\pi/2} R\cos\theta \cdot R \cdot d\theta}{L} = \frac{R^2}{\pi R/2} \cdot \int_0^{\pi/2} \cos\theta \cdot d\theta = \frac{2R}{\pi}$$

and in the similar manner we can find that

$$y_G = \frac{2R}{\pi}$$

It should be noted that $x_G = y_G$ since the structure (wire) has the axis of symmetry at 45°.

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Example 6.4 Find the location of a centroid of the quarter circle (Fig. 6.8).

Solution Quarter of a circle has an axis of symmetry at 45°. Thus, $x_G = y_G$. Let us use a horizontal strip located at the distance y and having thickness of dy. The equation of the circle is

$$x^2 + y^2 = R^2$$

and the limits defining the quarter circle are $0 < x < R$ and $0 < y < R$.

The area of the strip is $dA = x \cdot dy$, where

$$x = \sqrt{R^2 - y^2}$$

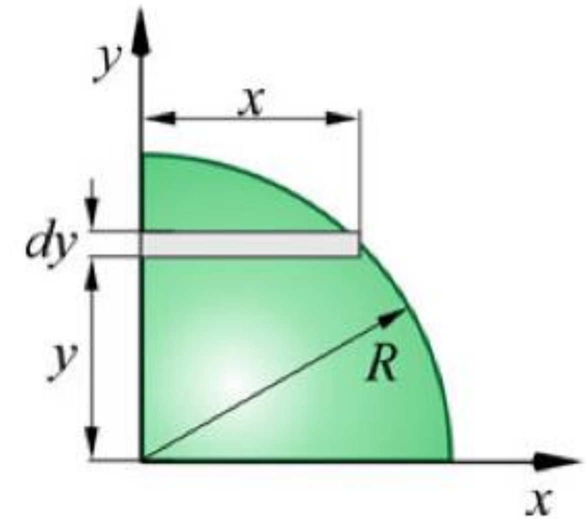

Fig. 6.8 Centroid of a quarter circle

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Thus, the vertical coordinate of the centroid of the quarter circle can be calculated by direct integration

$$y_G \cdot A = \int_0^R y \cdot dA = \int_0^R y \cdot x \cdot dy = \int_0^R y \cdot \sqrt{R^2 - y^2} \cdot dy = \frac{R^3}{3}$$

Since the area of the quarter circle is

$$A = \frac{\pi R^2}{4}$$

the location of the centroid in the y direction is

$$y_G = \frac{4R}{3\pi}$$

Due to the symmetry we have $x_G = \frac{4R}{3\pi}$.

6.3.2 Centroids of Composite Bodies

Determination of the centroids by integration may become a rather tedious task for complicated body shapes. Very often, an area may be divided into a number of basic elements, like rectangle, triangles, circles, and others, whose centroid coordinates may be easily obtained by using integration. The centroids of such elements are summarized in Table 6.1. Many structures are built from those simple shapes. For example, the structure in Fig. 6.9 is composed of two rectangles, a triangle, and a circular cutout. Their boundaries are shown as dotted lines. Since we know the centroid location of each constituent part, we may rewrite (6.3) and (6.4) utilizing (6.8) to get the centroid of the whole assembly.

$$x_G = \frac{\sum \Delta W_i \cdot x_i}{W} = \frac{\sum t \cdot \gamma \cdot \Delta A_i \cdot x_i}{t \cdot \gamma \cdot A} = \frac{\sum \Delta A_i \cdot x_i}{A} \qquad (6.15)$$

$$y_G = \frac{\sum \Delta W_i \cdot y_i}{W} = \frac{\sum t \cdot \gamma \cdot \Delta A_i \cdot y_i}{t \cdot \gamma \cdot A} = \frac{\sum \Delta A_i \cdot y_i}{A} \qquad (6.16)$$

Table 6.1 Centroids of basic elements

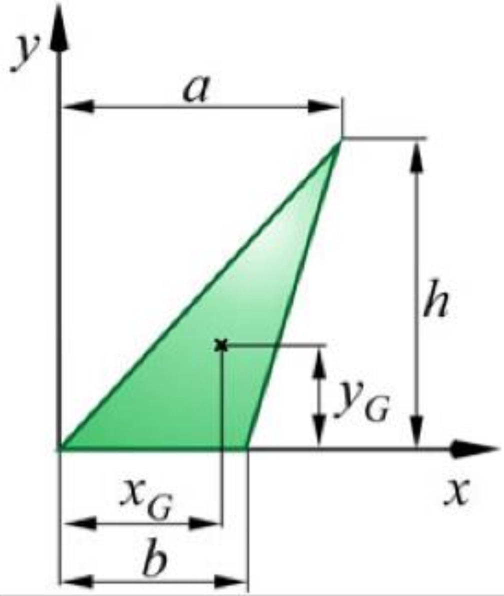
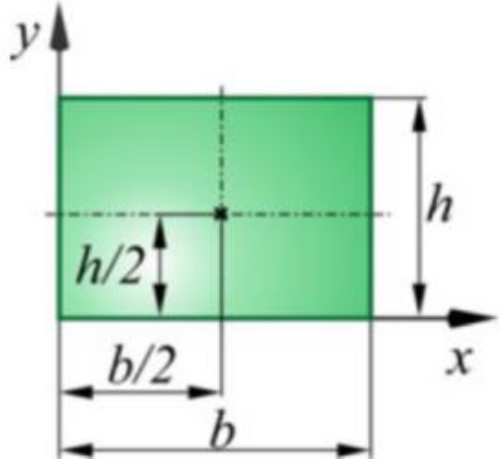
Element	Area	x_G	y_G
	$\frac{b \cdot h}{2}$	$\frac{a+b}{3}$	$\frac{h}{3}$
	$b \cdot h$	$\frac{b}{2}$	$\frac{h}{2}$

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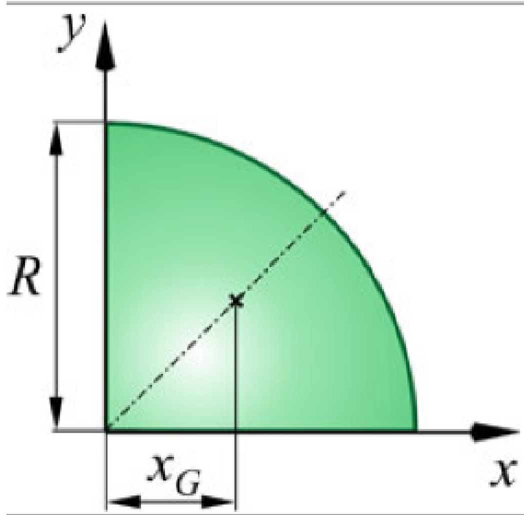
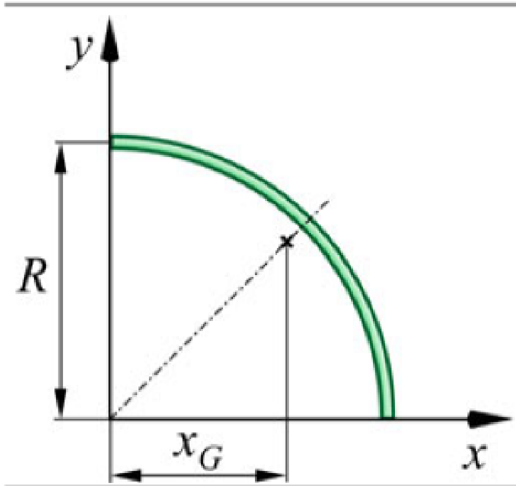
Table 6.1 Centroids of basic elements

Element	Area	x_G	y_G
	$\frac{\pi R^2}{4}$	$\frac{4R}{3\pi}$	$\frac{4R}{3\pi}$

Element	Length	x_G	y_G
	$\frac{\pi R}{2}$	$\frac{2R}{\pi}$	$\frac{2R}{\pi}$

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Guidelines and Recipes to Calculate the Centroid Location for Composite Bodies

- Select a common coordinate system.
- Divide the body into number of parts with simple geometry, so that for each part the location of its centroid is known.
- Identify the centroid location for each part in the common coordinate system and its corresponding area or length.

6.5.3 Theorems of Pappus

Many geometrical shapes, which are used in engineering practice, can be generated by revolving a plane curve or a flat surface about an axis. Greek mathematician Pappus of Alexandria (*circa* 290–350 CE) derived the theorem for calculating the surface area or volume created by revolving a plane curve or an area. They apply to the curves and areas that do not intersect the axis of rotation. He showed that the surface and the volume are related to the distance traveled by their centroids. These theorems are sometimes called Pappus–Guldinus. Guldinus (1577–1643) was a Swiss mathematician, who had rediscovered these theorems, but was not able to give a satisfactory proof of them.

Theorem 1 *The surface area created by revolution of a plane curve about the axis belonging to the same plane is equal to the length of the curve multiplied by the distance traveled by its centroid.*

The line AB is revolved about the x-axis (Fig. 6.20) by an angle of 2π. The differential length dl generates a surface of $2\pi y dl$. Thus, the entire area S generated after rotation of 2π is

$$S = \int_A^B 2\pi y dl = 2\pi \int_A^B y dl$$

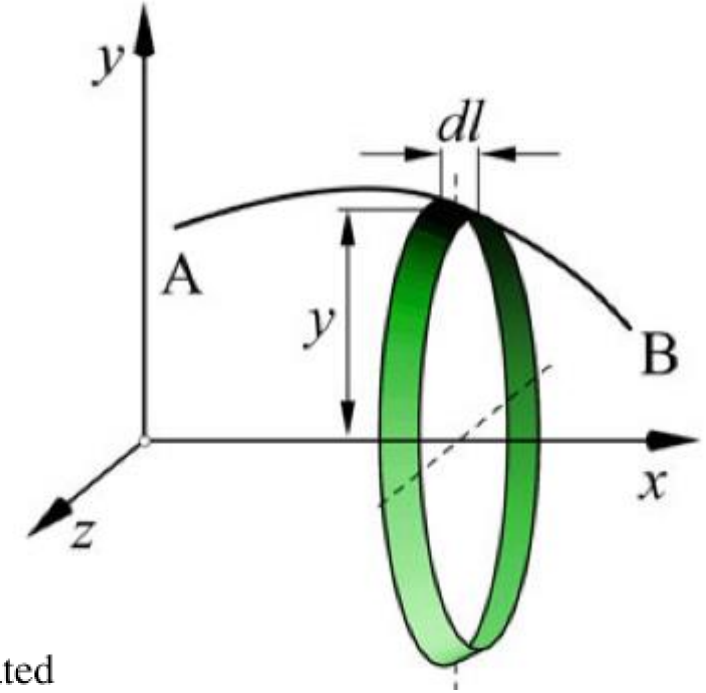

Fig. 6.20 Surface generated by a rotating line

As it was shown above, (6.12), the integral in the above equation is

$$\int_{\text{contour}} y dl = L \cdot y_{\mathrm{G}}$$

Therefore, we have

$$S = 2\pi L y_{\mathrm{G}} \tag{6.24}$$

where is the distances traveled by the centroid of the curve during the 2π revolution around the x-axis.

Theorem 2 *The volume of a body created by revolution of a plane area about the axis belonging to the same plane is equal to the area multiplied by the distance traveled by its centroid.*

Plane area A is revolving about the x-axis (Fig. 6.21) by an angle of 2π. Let's consider differential element dS. Rotation of element dS will generate a toroid with volume $dV = 2\pi y\, dS$. Thus, volume V generated by rotation of the whole area A by 2π is

$$V = \int_{\text{area}} 2\pi y\, dS = 2\pi \int_A y\, dS$$

The above integral can be represented as (see (6.12))

$$\int_A y\, dS = A y_G$$

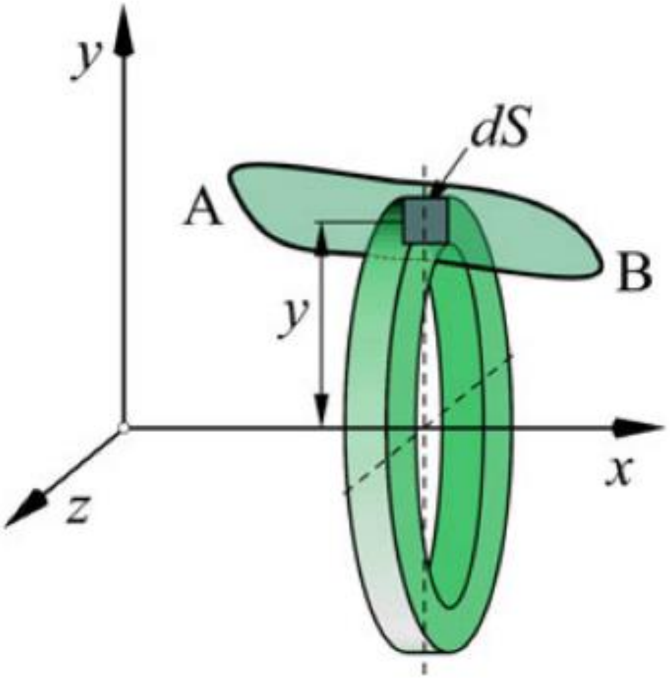

Fig. 6.21 Rotating a plane area

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where y_G is the centroid of area A. The volume of the body may be expressed as

$$V = 2\pi A y_G \tag{6.25}$$

Here, $2\pi y_G$ is the distance traveled by the centroid of the plane area.

If the curve or the area is revolved through an angle φ less than 2π, the resulted area or volume can be found by substituting 2π by the angle φ. Thus, (6.24) and (6.25) become

$$A = \varphi L y_G \tag{6.26}$$

and

$$V = \varphi A y_G \tag{6.27}$$