INTEGRAL INPUT-TO-STATE STABILITY OF TRAFFIC FLOW WITH VARIABLE SPEED LIMIT

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OUTLINE

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INTRODUCTION

MOTIVATION

- Variable Speed Limit (VSL): Tool in ITS to improve measures in traffic flow by changing the speed limit on a highway segment.

- Motivation: Saturated nature of VSL
- Upper Bound: VSL commands never exceed legal speed limit
- Lower Bound: VSL commands are bounded below by a certain operating limit
- Due to its dissipation characteristics, we consider to find a saturated control law which makes the closed-loop system (CLS) integral input-to-state stable (IISS)

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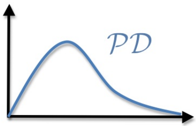
NOTATIONS: COMPARISON FUNCTIONS



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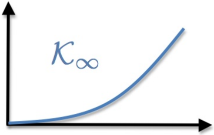

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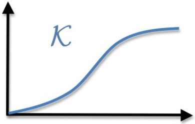

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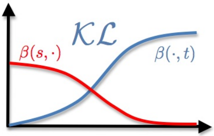

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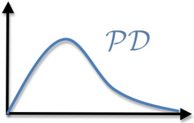

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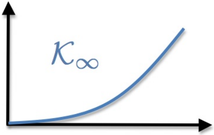

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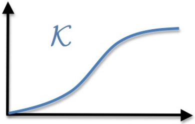

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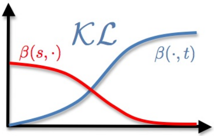

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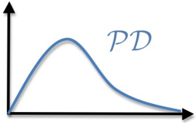

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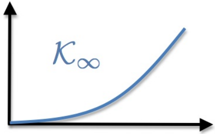

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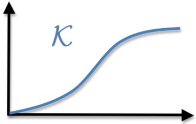

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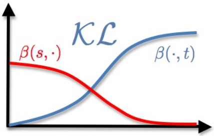

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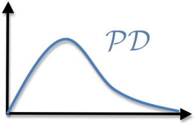

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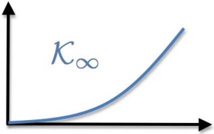

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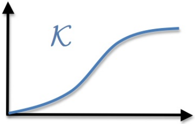

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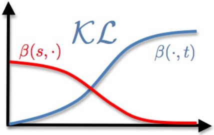

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Definition: Input-to-State Stability (ISS) (Sontag, IEEE TAC, 1989)

The system $\dot{x} = f(x, u)$ is ISS if there exist $\beta \in \mathcal{KL}$ and $\nu \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathbb{R}^n$ and all $u \in \mathcal{U}$,

$$|x(t; x_0, u)| \leq \beta(|x_0|, t) + \nu\left(\|u\|\right), \quad \forall t \geq 0.$$

- Vanishing transients "proportional" to initial state's norm
- Steady-state error "proportional" to input amplitude.

Definition: Integral Input-to-State Stability (iISS) (Sontag, SCL, 1998)

The system $\dot{x} = f(x, u)$ is iISS if there exist $\beta \in \mathcal{KL}$ and $\nu_1, \nu_2 \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathbb{R}^n$ and all $u \in \mathcal{U}$,

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Proposition: ISS and iISS characterization
(Sontag, Wang, SCL, 1995 & Angeli et al., IEEE TAC, 2000)

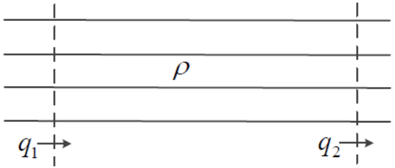
The system $\dot{x} = f(x, u)$ is ISS (resp. iISS) if and only if there exist a CLF V, $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$, $\gamma \in \mathcal{K}_\infty$, and $\alpha \in \mathcal{K}_\infty$ (resp. $\alpha \in \mathcal{PD}$) such that, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|)$$
$$\frac{\partial V}{\partial x} f(x, u) \leq -\alpha(|x|) + \gamma(|u|)$$

OUTLINE

PROBLEM DESCRIPTION

ODE MODEL OF TRAFFIC FLOW



Concerning the vehicle conservation, we have the following continuous-time ODE for a given road segment:

$$\dot{\rho} = \frac{1}{L}[q_1 - q_2], \tag{6}$$

- ρ: number of vehicles at a given length of the road segment,
- q_1: number of vehicles coming into the road segment,
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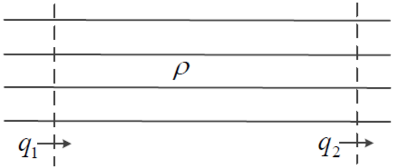

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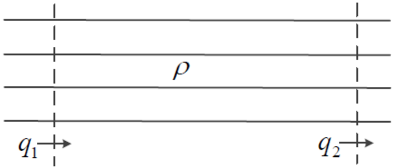

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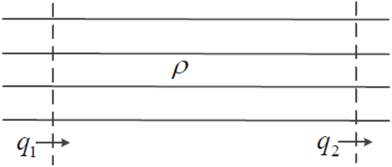

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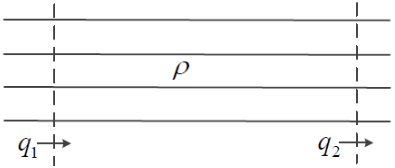

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- We use the triangular fundamental diagram for flow-density relationship.
- Speed-density relationship:

$$v(\rho) = \begin{cases} v_f & , \quad \rho < \rho_{cr} \\ C\left(\frac{1}{\rho} - \frac{1}{\rho_{max}}\right) & , \quad \rho \geq \rho_{cr} \end{cases}$$

  - v_f: free-flow speed,
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- Flow-density relationship can be obtained by $q = v\rho$:

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PROBLEM DESCRIPTION
VSL FORMULATION

- Define VSL rate $u_{VSL} : \mathbb{R}_{\geq 0} \to D \subset \mathbb{R}_{\geq 0}$ as follows:

$$v_f(u_{VSL}) = v_f^* \cdot u_{VSL} \tag{8}$$

  - v_f^*: free-flow speed of the non-VSL case.
- We have the following state space representation for (6) with the VSL formulation (8):

$$\dot{\rho} = \begin{cases} -\frac{v_f^*}{L}\rho \cdot u_{VSL} + \frac{1}{L}q_1 & , \ \rho < \rho_{cr} \\ -\frac{C^*}{L}\left(1 - \frac{\rho}{\rho_{max}}\right) \cdot u_{VSL} + \frac{1}{L}q_1 & , \ \rho \geq \rho_{cr} \end{cases} \tag{9}$$

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PROBLEM DESCRIPTION

VSL FORMULATION

- Define VSL rate $u_{VSL} : \mathbb{R}_{\geq 0} \to D \subset \mathbb{R}_{\geq 0}$ as follows:

$$v_f(u_{VSL}) = v_f^* \cdot u_{VSL} \tag{8}$$

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STATE FEEDBACK DESIGN

- We consider the state feedback controller of the form

$$u_{VSL}(\rho) = \begin{cases} 1 & , \ \rho < \rho_{cr} \\ \mathrm{sat}\left(\rho\left(1 - \frac{\rho}{\rho_{max}}\right); \frac{v_{min}}{v_f^*}, 1\right) & , \ \rho \geq \rho_{cr} \end{cases} \quad (10)$$

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- Note that the vector field of the CLS (11) is a piecewise defined function so that the results of [Sontag, Wang, SCL, 1995] and [Angeli et al., IEEE TAC, 2000] may not directly be applied.

Definition 1: IISS

The family of systems $\dot{x} = f_p(x, u)$, $\forall p \in \mathcal{P}$ is said to be IISS, if $\exists \alpha \in \mathcal{K}_\infty$, $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$, for every initial state $x(0)$, every input u and $p \in \mathcal{P}$ such that the estimate holds along all solutions:

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Proposition 4: IISS Lyapunov Characterization
(Liberzon, CDC, 1999 & Haimovich and Mancilla-Aguilar, Automatica, 2019)

The family of systems $\dot{x} = f_p(x, u)$, $\forall p \in \mathcal{P}$ is IISS if and only if $\exists V : \mathbb{R}^n \to \mathbb{R}$ and $\exists \alpha \in \mathcal{PD}$ and $\exists \alpha_1, \alpha_2, \gamma \in \mathcal{K}_\infty$ satisfying $\forall x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $p \in \mathcal{P}$:

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OUTLINE

MAIN RESULT

- The CLS:

$$\dot{\rho} = \begin{cases} -\frac{v_f^*}{L}\rho + \frac{1}{L}q_1 & , \ \rho < \rho_{cr} \\ -\left(\frac{C^*}{L}\right)\left(1 - \frac{\rho}{\rho_{max}}\right) \operatorname{sat}\left(\rho\left(1 - \frac{\rho}{\rho_{max}}\right)\right) + \frac{1}{L}q_1 & , \ \rho \geq \rho_{cr}. \end{cases} \quad (11)$$

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Proof (Sketch):

Define $V(\rho) = \ln(1+\rho^2)$. Let us consider two cases:

- **Case 1:** Consider $\rho < \rho_{cr}$. Deriving V along soln's of (11) yields

$$\dot{V}|_{(11)} = \frac{2\rho\dot{\rho}}{1+\rho^2} = -\frac{(2v_f^*/L)\rho^2}{1+\rho^2} + \frac{1}{L}\frac{2\rho}{1+\rho^2}q_1 \le -\eta_1(|\rho|) + \gamma(|q_1|). \quad (13)$$

where $\eta_1(s) = \frac{(2v_f^*/L)s^2}{1+s^2}$ and $\gamma(s) = (1/L)|s|$ by observing $\left|\frac{2\rho}{1+\rho^2}\right| \le 1$.
One can see that $\eta_1 \in \mathcal{PD}$ and $\gamma \in \mathcal{K}_\infty$.

- **Case 2:** Consider $\rho \ge \rho_{cr}$. Deriving V soln's of (11), we have

$$\begin{aligned}\dot{V}|_{(11)} =& \frac{2\rho\dot{\rho}}{1+\rho^2} \\ =& -\frac{\left(\frac{2C^*}{L}\right)\rho\left(1-\frac{\rho}{\rho_{max}}\right) sat\left(\rho\left(1-\frac{\rho}{\rho_{max}}\right)\right)}{1+\rho^2} + \frac{1}{L}\frac{2\rho}{1+\rho^2}q_1 \quad (15) \\ \le& -\eta_2(|\rho|) + \gamma(|q_1|)\end{aligned}$$

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NUMERICAL EXAMPLES

EXAMPLE 1

We choose

$$L = 1\ km,\ \rho(0) = 0\ veh/km,\ \rho_{max} = 200\ veh/km,$$
$$\rho_{cr} = 20\ veh/km,\ v_f^* = 100\ km/h,\ v_{min} = 50\ km/h, \quad (17)$$

and

$$q_1(t) = q_{1,cap} \cdot exp((t - t_{peak})^2/2\sigma)$$

with $q_{1,cap} = 1000\ veh/h$, $t_{peak} = 10\ h$ and $\sigma = 10$ to introduce a peak demand scenario for the road segment. One can see that q_1 satisfies the bounded energy assumption:

$$\int_0^{2t_{peak}} q_1(\tau)d\tau \leq \int_{-\infty}^{\infty} q_1(\tau)d\tau \leq q_{1,cap}\sqrt{2\pi}\sigma < \infty$$

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- VSL: $$u_{VSL}(\rho) = \begin{cases} 1 & , \ \rho < \rho_{cr} \\ \mathrm{sat}\left(\rho\left(1 - \frac{\rho}{\rho_{max}}\right); \frac{v_{min}}{v_f^*}, 1\right) & , \ \rho \geq \rho_{cr} \end{cases} \tag{10}$$

NUMERICAL EXAMPLES

EXAMPLE 1

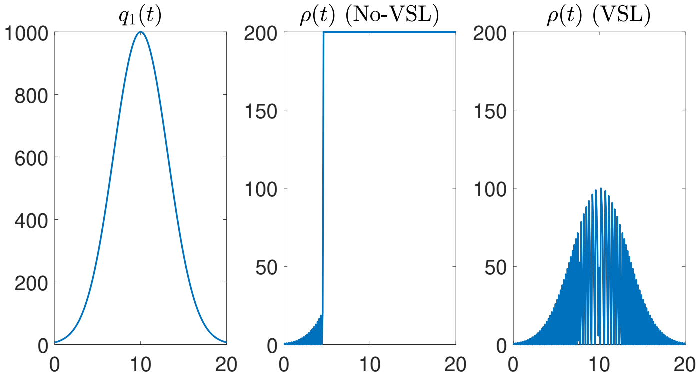

Figure: Simulation Results for No-VSL and VSL Cases under the Input satisfying Bounded Energy Assumption.

NUMERICAL EXAMPLES

EXAMPLE 2

We choose

$$L = 1\ km,\ \rho(0) = 0\ veh/km,\ \rho_{max} = 200\ veh/km,$$
$$\rho_{cr} = 20\ veh/km,\ v_f^* = 100\ km/h,\ v_{min} = 50\ km/h,\ . \quad (17)$$

As a second example, we select

$$q_1(t) = \frac{q_{1,cap}}{((t - t_{peak})^2 + 1)}$$

with $q_{1,cap} = 1000\ veh/h$, $t_{peak} = 10\ h$ and $\sigma = 10$ to introduce a peak demand scenario for the road segment. One can see that q_1 satisfies the bounded energy assumption:

$$\int_0^{2t_{peak}} q_1(\tau)d\tau \leq \int_{-\infty}^{\infty} q_1(\tau)d\tau \leq q_{1,cap}\sqrt{2\pi}\sigma < \infty$$

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NUMERICAL EXAMPLES

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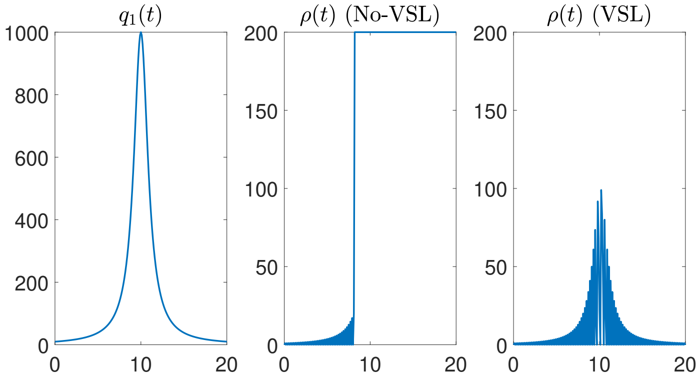

Figure: Simulation Results for No-VSL and VSL Cases under the Input satisfying Bounded Energy Assumption.

OUTLINE

CONCLUSIONS

- Conclusions:
  - We introduced a VSL controller to ensure IISS of the traffic state by using saturated feedback.
  - Two-phase fundamental diagram implementation $\Rightarrow$ state-dependent switched ordinary differential equation.
  - IISS of CLS was guaranteed by a common Lyapunov function.
  - Some robustness properties were demonstrated by numerical examples
- Future Studies:
  - Obtain the conditions for multiple segments
    - Small-gain results to preserve IISS in cascade systems [Chaillet, Angeli, SCL, 2008].
  - Show the validity of the results through micro-simulation.
  - Implement other phenomena such as lane closure, mixed traffic scenarios, etc.

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