Analysis of Integral Input-to-State Stable time-delay systems in cascade

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3 Illustrative Example
\[
\begin{align*}
\{ & \alpha \text{ continuous } \\
& \alpha(0) = 0 \\
& \alpha(s) > 0, \forall s > 0 \\
\} \\
\alpha \in \mathcal{PD} \\
\alpha \text{ increasing}
\end{align*}
\]

\[
\{ & \alpha \in \mathcal{K} \\
& \lim_{s \to \infty} \alpha(s) = \infty \\
\} \\
\beta(\cdot, t) \in \mathcal{K}, \forall t \geq 0 \\
\beta(s, \cdot) \text{ nonincreasing, } \forall s \geq 0 \\
\lim_{t \to \infty} \beta(s, t) = 0, \forall s \geq 0
\]
Consider the nonlinear TDS: \( \dot{x}(t) = f(x_t, u(t)) \)

- State History: \( x_t \in C^n \) defined with the maximum delay \( \delta \geq 0 \) as 
  \[
  x_t(s) := x(t + s), \quad \forall s \in [-\delta, 0].
  \]

- \( C \): Set of all continuous functions \( \varphi : [-\delta; 0] \rightarrow \mathbb{R} \).
- \( U \): Set of measurable essentially bounded signals to \( \mathbb{R}^m \).
- Given \( x \in \mathbb{R}^n \), \( |x| \) denotes its Euclidean norm.
- Given any \( \phi \in C^n \), \( \|\phi\| := \sup_{\tau \in [-\delta, 0]} |\phi(\tau)| \).
- \( f : C^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \), Lipschitz on bounded sets and to satisfy \( f(0, 0) = 0 \).
Lyapunov-Krasovskii functional (LKF) candidate: Any functional $V : C^n \to \mathbb{R}_{\geq 0}$, Lipschitz on bounded sets, for which there exist $\alpha, \bar{\alpha} \in \mathcal{K}_\infty$ such that

$$\alpha(|\phi(0)|) \leq V(\phi) \leq \bar{\alpha}(\|\phi\|), \quad \forall \phi \in C^n. \quad (6)$$

Its upper-right Dini derivative along the solutions of $\dot{x}(t) = f(x_t, u(t))$ is then defined for all $t \geq 0$ as

$$D^+ V(x_t, u(t)) := \limsup_{h \to 0^+} \frac{V(x_{t+h}) - V(x_t)}{h}. \quad (7)$$
Definition (0-GAS)

The TDS is said to be **globally asymptotically stable in the absence of inputs** (0-GAS) if there exists $\beta \in \mathcal{KL}$ such that, the solution of the input-free system $\dot{x}(t) = f(x_t, 0)$ satisfies

$$|x(t)| \leq \beta(\|x_0\|, t), \quad \forall t \geq 0.$$ 

Definition (iISS, (Pepe, Jiang, SCL, 2006))

The TDS is said to be **integral input-to-state stable (iISS)** if there exists $\beta \in \mathcal{KL}$ and $\nu, \sigma \in \mathcal{K}_\infty$ such that, its solution satisfies

$$|x(t)| \leq \beta(\|x_0\|, t) + \nu \left( \int_0^t \sigma(|u(s)|)ds \right), \quad \forall t \geq 0.$$

- Forward completeness (Hale, 1977, Theorem 3.2, p. 43)
- Asymptotic stability in the absence of inputs (0-GAS)
Definition (BEBS, BECS)

The TDS is said to have the **bounded energy-bounded state** (BEBS) property, if there exists $\zeta \in \mathcal{K}_\infty$ such that its solution satisfies

$$\int_0^\infty \zeta(|u(s)|) ds < \infty \quad \Rightarrow \quad \sup_{t \geq 0} |x(t)| < \infty.$$  

It is said to have the **bounded energy-converging state** (BECS) property if there exists $\zeta \in \mathcal{K}_\infty$ such that, its solution satisfies

$$\int_0^\infty \zeta(|u(s)|) ds < \infty \quad \Rightarrow \quad \lim_{t \to \infty} |x(t)| = 0.$$  

Proposition (iISS $\Rightarrow$ 0-GAS, BEBS, BECS)

*If the TDS is iISS, then it is BEBS and BECS.*
Proposition (iISS LKF, Necessity: (Lin, Wang, CDC, 2018), Sufficiency: (Pepe, Jiang, SCL, 2006))

The TDS is iISS if and only if there exists a LKF candidate $V : \mathbb{C}^n \rightarrow \mathbb{R}_{\geq 0}$, $\alpha \in \mathcal{P}\mathcal{D}$ and $\gamma \in \mathcal{K}_{\infty}$, such that the following holds:

$$D^+ V(x_t, u(t)) \leq -\alpha(V(x_t)) + \gamma(|u(t)|), \quad \forall t \geq 0.$$  

→ Finite-dimensional case: (Angeli et al., IEEE TAC, 2000).

Proposition (Sufficient Condition for iISS, (Chaillet, Pepe, CDC, 2018))

The TDS is iISS if there exists a LKF candidate $V : \mathbb{C}^n \rightarrow \mathbb{R}_{\geq 0}$, $\alpha \in \mathcal{P}\mathcal{D}$ and $\eta, \gamma \in \mathcal{K}_{\infty}$, such that the following holds:

$$D^+ V(x_t, u(t)) \leq -\frac{\alpha(|x(t)|)}{1 + \eta(\|x_t\|)} + \gamma(|u(t)|), \quad \forall t \geq 0.$$
Consider two nonlinear TDS in cascade:

\[
\Sigma_1 \delta : \quad \dot{x}_1(t) = f_1(x_{1t}, x_{2(t - \delta_1)}), \tag{9a}
\]

\[
\Sigma_2 \delta : \quad \dot{x}_2(t) = f_2(x_{2t}, u(t)), \tag{9b}
\]

\[\rightarrow \delta_1 \in [0, \delta]: \text{Interconnection through discrete delay.}\]

**Questions:**

- iISS preserved under cascade interconnected TDS?
- If not, conditions to ensure iISS?
- Conditions to ensure 0-GAS and BEBS?
Consider two nonlinear systems in cascade:

\[ \Sigma_1 : \dot{x}_1 = f_1(x_1, x_2) \]
\[ \Sigma_2 : \dot{x}_2 = f_2(x_2, u) . \]

- **ISS** is naturally preserved in cascade [Sontag, EJC, 1995]
- **iISS** is **not** preserved by cascade [Panteley, Loría, Automatica, 2001], [Arcak et al., SIAM JCO, 2002].

**Questions:**
- **iISS** preserved under cascade interconnected TDS?
- If not, conditions to ensure iISS?
- Conditions to ensure 0-GAS and BEBS?
Theorem (Chaillet, Angeli, SCL, 2008)

Let $V_1$ and $V_2$ be two Lyapunov functional candidates. Assume that there exist $\gamma_1, \gamma_2 \in K$, and $\alpha_1, \alpha_2 \in PD$ such that, for all $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and all $u \in \mathbb{R}^m$,

\[
\frac{\partial V_1}{\partial x_1}(x_1)f_1(x_1, x_2) \leq -\alpha_1(|x_1|) + \gamma_1(|x_2|)
\]

\[
\frac{\partial V_2}{\partial x_2}(x_2)f_2(x_2, u) \leq -\alpha_2(|x_2|) + \gamma_2(|u|).
\]

$\rightarrow q_2(s) = O_{s \to 0^+}(q_1(s))$: Given $q_1, q_2 \in PD$, we say that $q_1$ has greater growth than $q_2$ around zero if $\exists k \geq 0$ such that $\limsup_{s \to 0^+} q_2(s)/q_1(s) \leq k$.

Questions:

- If not, conditions to ensure iISS?
- Conditions to ensure 0-GAS and BEBS?

Above condition valid for TDS?
Theorem

Assume that \( \exists \) two LKF candidates \( V_i : C^n_i \to \mathbb{R}_{\geq 0} \) and \( \eta_i \in K_{\infty}, i \in \{1, 2\} \), such that the following holds along any solution of \( \dot{x}_1(t) = f_1(x_1, u_1(t)) \)

\[
D^+ V_1(x_1, u_1(t)) \leq -\frac{\alpha_1(|x_1(t)|)}{1 + \eta_1(V_1(x_1))} + \gamma_1(|u_1(t)|) \tag{10}
\]

and the following holds along any solution of \( \dot{x}_2(t) = f_2(x_2, u(t)) \)

\[
D^+ V_2(x_2, u(t)) \leq -\frac{\alpha_2(|x_2(t)|)}{1 + \eta_2(V_2(x_2))} + \gamma_2(|u(t)|) \tag{11}
\]

for all \( t \geq 0 \).

\[
\gamma_1(s) = O_{s \to 0^+}(\alpha_2(s)). \tag{12}
\]

Then, the cascade is 0-GAS and satisfies the BEBS property.
Lemma

Let $V : \mathcal{C}^n \to \mathbb{R}_{\geq 0}$ be a LKF candidate satisfying, along any solution of the TDS $\dot{x}(t) = f(x_t)$,

$$D^+ V(x_t) \leq -\frac{\alpha(|x(t)|)}{1 + \eta(V(x_t))},$$

for some $\alpha \in \mathcal{P}\mathcal{D}$ and $\eta \in \mathcal{K}_\infty$. Let $\tilde{\alpha} \in \mathcal{P}\mathcal{D}$ satisfying

$$\tilde{\alpha}(s) = \mathcal{O}_{s \to 0^+}(\alpha(s)).$$

Then, $\exists$ a continuously differentiable function $\rho \in \mathcal{K}_\infty$ such that the functional $\tilde{V} := \rho \circ V$ satisfies

$$D^+ \tilde{V}(x_t) \leq -\tilde{\alpha}(|x(t)|).$$
Proof of Lemma (Sketch).

- Take continuous non-decreasing function $q : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ satisfying $q(s) > 0$ for all $s > 0$ such that $\rho$ can be written as $\rho(s) = \int_0^s q(r) dr$ for all $s \geq 0$ and choose $\tilde{V} = \rho \circ V$.

- Its Dini derivative along the solutions of $\dot{x}(t) = f(x_t)$ reads

$$
D^+ \tilde{V}(x_t) \leq -q(V(x_t)) \frac{\alpha(|x(t)|)}{1 + \eta(V(x_t))}.
$$

- Define $\mu : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ as $\mu(s) := \sup_{r \in [0,s]} \frac{\tilde{\alpha}(r)}{\alpha(r)}, \forall s \geq 0$.

- (15) ensures the boundedness of $\mu$ on $[0, a], a > 0$.

- Choose $q(s) := \mu \circ \alpha^{-1}(s)(1 + \eta(s)), \forall s \geq 0$.

- Then, we have

$$
q(V(x_t)) \frac{\alpha(|x(t)|)}{1 + \eta(V(x_t))} \geq \mu \circ \alpha^{-1}(V(x_t))\alpha(|x(t)|)
\geq \mu(|x(t)|)\alpha(|x(t)|) \geq \tilde{\alpha}(|x(t)|).
$$
Proof of Theorem: Forward Completeness.

- (11) implies forward completeness of $\dot{x}_2(t) = f_2(x_{2t}, u(t))$.
- (10) with $u_1(t) = x_2(t - \delta_1) \Rightarrow \nexists$ any finite escape time for $x_1(t)$.

Proof of Theorem: 0-GAS (Sketch).

- Consider the input-free system
  \begin{align}
  \dot{x}_1(t) &= f_1(x_{1t}, x_2(t - \delta_1)), \\
  \dot{x}_2(t) &= f_2(x_{2t}, 0). 
  \end{align}

- (12)+Lemma $\Rightarrow \exists \rho \in \mathcal{K}_\infty \cap C^1$ such that $\tilde{V}_2 := \rho \circ V_2$ satisfies
  \begin{equation}
  D^+ \tilde{V}_2(x_{2t}) \leq -2\gamma_1(||x_2(t)||).
  \end{equation}

- Now, consider the LKF defined as
  \begin{equation}
  V_2(\phi_2) := \tilde{V}_2(\phi_2) + \int_{-\delta_1}^{0} \gamma_1(||\phi_2(\tau)||) d\tau, \quad \forall \phi_2 \in C^{n_2}.
  \end{equation}
Proof of Theorem: 0-GAS (Sketch-Continued).

- In view of (21), its Dini derivative therefore reads

$$D^+ V_2(x_{2t}) \leq -\gamma_1(|x_2(t)|) - \gamma_1(|x_2(t - \delta_1)|).$$

- Furthermore (10) ensures that

$$D^+ V_1(x_{1t}, x_2(t-\delta_1)) \leq -\frac{\alpha_1(|x_1(t)|)}{1 + \eta_1(V_1(x_{1t}))} + \gamma_1(|x_2(t - \delta_1)|).$$

- Summing this with (24), we get that

$$D^+ \mathcal{V}(x_t) \leq -\frac{\alpha_1(|x_1(t)|) + \gamma_1(|x_2(t)|)}{1 + \eta_1(\mathcal{V}(x_t))},$$

□
Proof of Theorem: BEBS.

- (11) ⇒ ∃β_2 ∈ ℜL, ν_2, σ_2 ∈ ℜ∞ such that,

  \[ |x_2(t)| \leq \beta_2(\|x_20\|, t) + \nu_2 \left( \int_0^t \sigma_2(|u(s)|) ds \right), \quad \forall t \geq 0. \quad (28) \]

- Assume that the following bounded energy holds for some \( c \geq 0 \).

  \[ \int_0^\infty \max \{\gamma_2(|u(\tau)|), \sigma_2(|u(\tau)|)\} d\tau \leq c \quad (29) \]

- Then, we ensure that \( \lim_{t \to \infty} |x_2(t)| = 0 \) and \( \exists T := T_{x_20, u} \geq 0 \) such that \( \|x_2(t)\| \leq 1, \forall t \geq T \), which guarantees that \( V_2(x_2(t)) \leq \bar{\alpha}_2(1), \forall t \geq T \).

- Integrating the dissipation inequality (11) of \( V_2 \), we have, for all \( t \geq T \),

  \[
  V_2(x_2(t)) - V_2(x_20) \leq - \int_0^t \frac{\alpha_2(|x_2(\tau)|)}{1 + \eta_2(\|V_2(x_2(\tau))\|)} d\tau + \int_0^t \gamma_2(|u(\tau)|) d\tau \\
  \leq - \int_T^\infty \frac{\alpha_2(|x_2(\tau)|)}{\bar{\eta}_2} d\tau + \int_0^\infty \gamma_2(|u(\tau)|) d\tau, 
  \]

where \( \bar{\eta}_2 := 1 + \eta_2 \circ \bar{\alpha}_2(1) \).
Proof of Theorem: BEBS (Continued).

- From (29), \( \int_T^\infty \alpha_2(\|x_2(\tau)\|)d\tau \leq (\alpha_2(\|x_{20}\|) + c) \bar{\eta}_2 \).
- From growth rate condition (12), \( \exists k > 0 \) s.t. \( \gamma_1(s) \leq k \alpha_2(s) \) \( \forall s \in [0, 1] \).
- It follows that
  \[
  \int_{-\delta_1}^\infty \gamma_1(\|x_2(\tau)\|)d\tau \leq \int_{-\delta_1}^T \gamma_1(\|x_2(\tau)\|)d\tau + \int_T^\infty k \alpha_2(\|x_2(\tau)\|)d\tau.
  \]
- Integrating dissipation inequality (10) with \( u_1(t) = x_2(t - \delta_1) \), we have
  \[
  \alpha_1(\|x_1(t)\|) \leq \alpha_1(\|x_{10}\|) + \int_0^t \gamma_1(\|x_2(\tau - \delta_1)\|)d\tau
  \leq \alpha_1(\|x_{10}\|) + \int_{-\delta_1}^{t-\delta_1} \gamma_1(\|x_2(\tau)\|)d\tau.
  \]
- It holds that
  \[
  \alpha_1(\|x_1(t)\|) \leq \alpha_1(\|x_{10}\|) + \int_0^T \gamma_1(\|x_2(\tau)\|)d\tau + \tilde{c}(\|x_0\|).
  \]
- The cascade owns the BEBS property. \( \square \)
The growth rate condition $\gamma_1(s) = O_{s \to 0^+}(\alpha_2(s))$ is reminiscent of the one obtained in [Chaillet, Angeli, SCL, 2008].

In [Chaillet, Angeli, SCL, 2008], it was shown that the growth rate condition implies iISS in finite-dimensional systems. This is due to the fact that, 0-GAS+(a relaxed version of) BEBS implies iISS in finite-dimensional systems as presented in [Angeli et al., SIAM JCO, 2004].

Not yet been extended to TDS.

The small-gain results for interconnected iISS TDS in [Ito et. al., Automatica, 2010]

- involves the upper and lower bounds on $V_1$ and $V_2$, thus leading to a more conservative condition,
- imposes that the dissipation rates for the driving and driven subsystems are of class class $\mathcal{K}$ (rather than $\mathcal{PD}$), meaning that both subsystems are required to have an ISS-like behavior for small inputs and
- cannot be used for our illustrative example.
Consider the following input-free cascade:

\[
\begin{align*}
\dot{x}_1(t) &= f_1(x_{1t}, x_2(t - \delta_1)) \\
\dot{x}_2(t) &= f_2(x_{2t}).
\end{align*}
\]  

(13a)  

(13b)

**Corollary**

Assume that there exist two LKF candidates \( V_1 : C^{n_1} \rightarrow \mathbb{R}_{\geq 0} \) and \( V_2 : C^{n_2} \rightarrow \mathbb{R}_{\geq 0}, \alpha_i, \overline{\alpha}_i, \eta_i \in K_\infty, \alpha_i \in K, i \in \{1, 2\}, \) and \( \gamma_1 \in K_\infty \) such that, the following holds along any solution of \( \dot{x}_1(t) = f_1(x_{1t}, u_1(t)) \)

\[
D^+ V_1(x_{1t}, u_1(t)) \leq -\frac{\alpha_1(\|x_1(t)\|)}{1 + \eta_1(V_1(x_{1t}))} + \gamma_1(\|u_1(t)\|),
\]

and the following holds along any solution of (13b)

\[
D^+ V_2(x_{2t}) \leq -\frac{\alpha_2(\|x_2(t)\|)}{1 + \eta_1(V_2(x_{2t}))}, \quad \forall t \geq 0.
\]

Assume also that \( \gamma_1(s) = O_{s \rightarrow 0^+}(\alpha_2(s)). \)
Example

Consider the following cascade TDS:

\[
\begin{align*}
\dot{x}_1(t) &= -\text{sat}(x_1(t)) + \frac{1}{4}\text{sat}(x_1(t-1)) + x_1(t)x_2(t-2)^2 \tag{34a} \\
\dot{x}_2(t) &= -\frac{3}{2}x_2(t) + x_2(t-1) + u(t) \int_{t-1}^{t} x_2(\tau)d\tau. \tag{34b}
\end{align*}
\]

\[\text{sat}(s) := \text{sign}(s) \min\{|s|, 1\} \text{ for all } s \in \mathbb{R}.\]

\[n_1 = n_2 = 1, \ m = 1, \ \delta_1 = \delta = 2.\]

Consider the LKF candidates defined as

\[
\begin{align*}
V_1(\phi_1) &= \ln \left(1 + \phi_1(0)^2 + \frac{1}{2} \int_{-1}^{0} \phi_1(\tau)\text{sat}(\phi_1(\tau))d\tau \right), \tag{35a} \\
V_2(\phi_2) &= \ln \left(1 + \phi_2(0)^2 + \int_{-1}^{0} \phi_2(\tau)^2d\tau \right), \tag{35b}
\end{align*}
\]
By deriving, we have

\[ D^+ V_1(x_{1t}, x_{2t}) \leq -\frac{x_1(t)\text{sat}(x_1(t))}{1 + \eta_1(V_1(x_{1t}))} + 2x_2(t - 2)^2, \]

\[ D^+ V_2(x_{2t}, u(t)) \leq -\frac{x_2(t)^2}{1 + \eta_2(V_2(x_{2t}))} + |u(t)|. \]

where \( \eta_1(s) = \eta_2(s) = e^s - 1 \). The functions are

- \( \alpha_1(s) = \text{sat}(s)s \),
- \( \alpha_2(s) = s^2 \),
- \( \eta_1(s) = \eta_2(s) = e^s - 1 \),
- \( \gamma_1(s) = 2s^2 \) and
- \( \gamma_2(s) = s \).

→ Growth-rate condition: \( 2s^2 = O_{s \to 0^+}(s^2) \).

The assumptions of Theorem are fulfilled. Thus, the cascade (35) is 0-GAS and owns the BEBS property.
Conditions under which the cascade of two iISS TDS is 0-GAS and has the BEBS property.

Growth restrictions on the input rate of the driven subsystem and the dissipation rate of the driving one.

An academic example illustrates the applicability of the result.

Limitations:
- More generic interconnection of the form $\dot{x}_1(t) = f_1(x_{1t}, x_{2t})$.
- Concluding that the overall cascade is iISS.
  - 0-GAS+BEBS$\Rightarrow$iISS for TDS?
- Allowing the input $u$ to impact directly the driven subsystem.
Acknowledgement
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