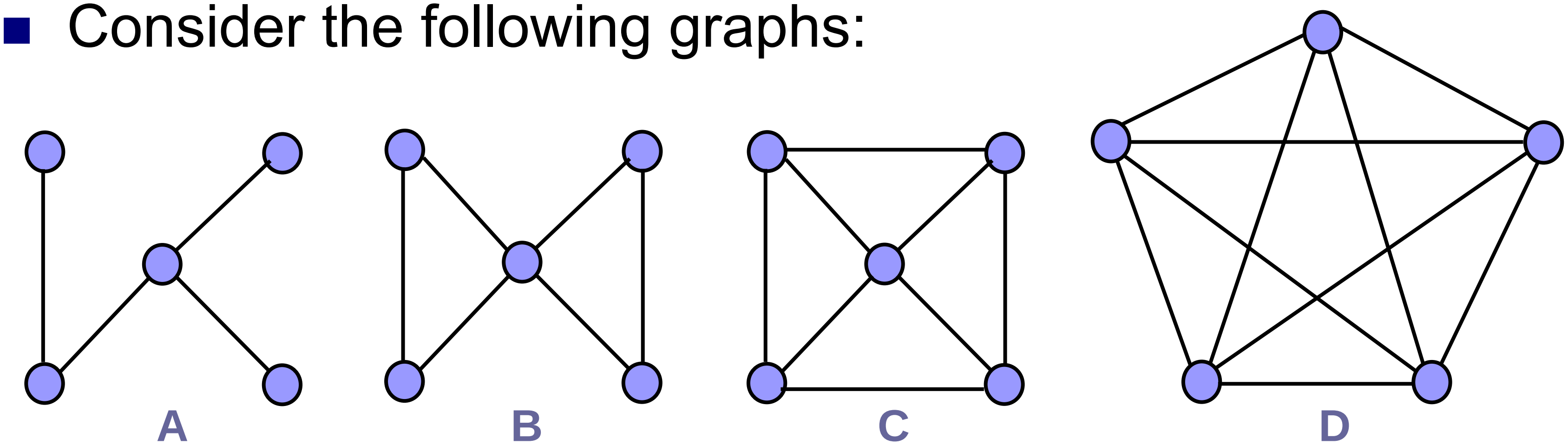
GRAPH THEORY and APPLICATIONS

Connectivity

Connectivity

- Consider the following graphs:



- A is a tree. Deleting any edge disconnects it.
- B cannot be disconnected by deleting single edge, but can be disconnected by deleting one vertex.
- C does not have any cut edge or cut vertex.
- D is still more connected than C.
- Intuitively each graph is more strongly connected than the previous one.

Vertex Cut

- Vertex cut: A subset V' of V such that G – V' is disconnected.
- k-vertex cut: A vertex cut of k elements.
  - A complete graph has no vertex cut.
- The connectivity $\kappa(G)$ is:
  - If G has at least one pair of non-adjacent vertices, minimum k for which G has a k-vertex cut.
  - Otherwise, $\kappa(G) = v - 1$
- $\kappa(G)=0$ if G is disconnected.
- G is k-connected if $\kappa(G) \geq k$.
  - All connected graphs with $v > 1$ are 1-connected.

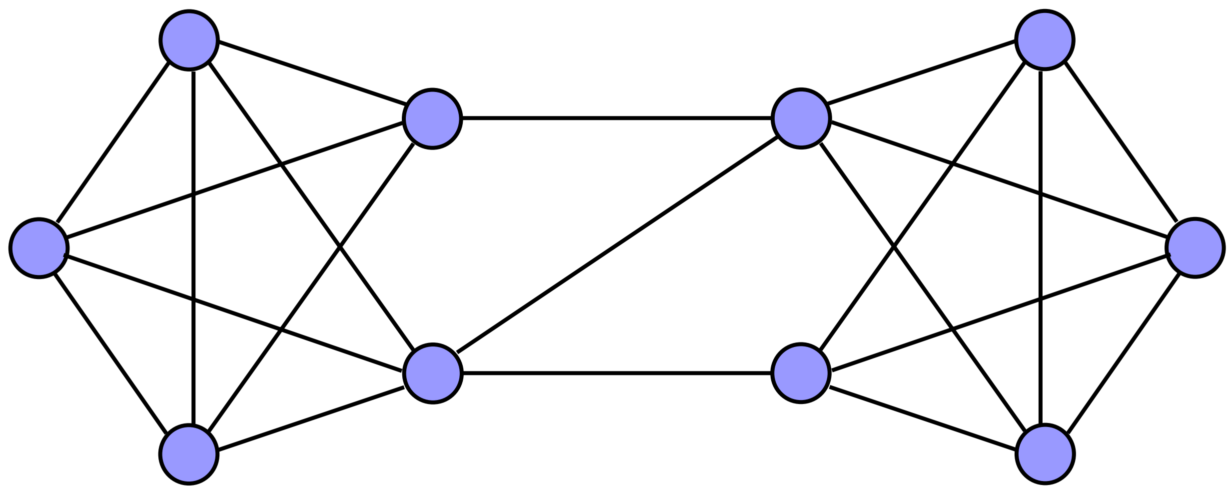
Edge Cut

- Edge cut: A subset of E of the form $[S, \overline{S}]$ where S is a nonempty, proper subset of V.
- k-edge cut: An edge cut of k elements.
- The edge-connectivity $\kappa'(G)$ is:
  - If G has at least one pair of vertices, minimum k for which G has a k-edge cut.
- $\kappa'(G)=0$ if G is disconnected or $v = 1$.
- G is k-edge-connected if $\kappa'(G) \geq k$.
  - All connected graphs with $v > 1$ are 1-edge-connected.

Connectivity

Theorem: $\kappa \leq \kappa' \leq \delta$

- The inequalities are often strict.



$\kappa = 2$
$\kappa' = 3$
$\delta = 4$

Connectivity pair

- Separating a graph by removing a mixed set of vertices and edges.

Connectivity pair:

- An ordered pair (a,b) of nonnegative integers, such that there is:
  - a set of a vertices, and
  - a set of b edges

  whose removal disconnects the graph.
- There is no:
  - set of a-1 vertices and b edges, or
  - set of a vertices and b-1 edges

  with this property.

Connectivity pair

- The two ordered pairs $(\kappa,0)$ and $(0,\kappa')$ are connectivity pairs.
- The connectivity pair generalizes both vertex and edge connectivity.
- For each value of a, $0 \leq a \leq \kappa$ there is a unique connectivity pair (a,b_a).
  - G has exactly $\kappa + 1$ connectivity pairs.

Connectivity function

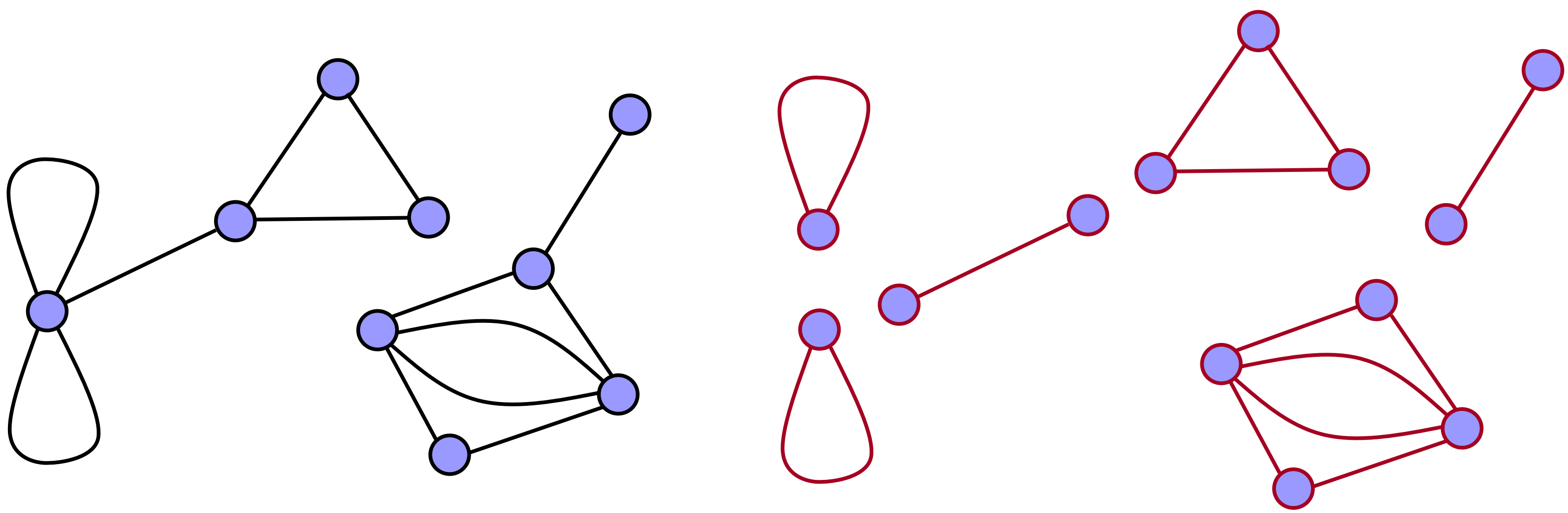
- The connectivity pairs of a graph G determine a function f,
  - from the set of $\{1, 2, \ldots, \kappa\}$
  - into the nonnegative integers

  such that $f(\kappa) = 0$.
- The connectivity function is strictly decreasing.

Theorem: Every decreasing function f from $\{1, 2, \ldots, \kappa\}$ into the nonnegative integers, such that $f(\kappa) = 0$, is the connectivity function of some graph.

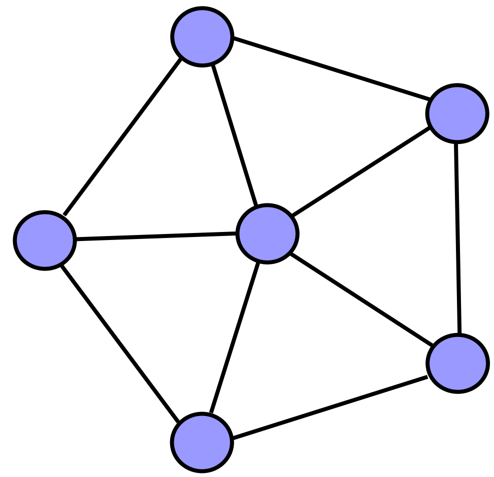
Blocks

- Block: A connected graph that has no cut vertex.
  - A block with $v \geq 3$ is 2-connected.
- Block of a graph: A subgraph that is:
  - a block
  - maximal with respect to this property.
- Every graph is the union of its blocks.



Characterization of 3-connected graphs

The wheel: For n≥4, W_n is defined to be the graph: K_1+C_{n-1}

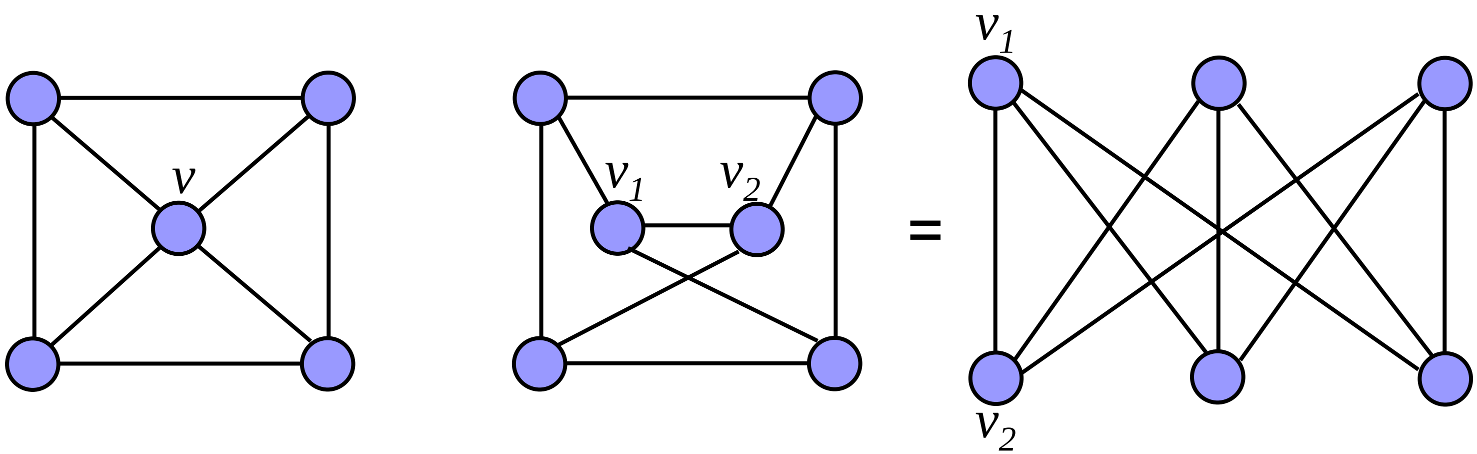


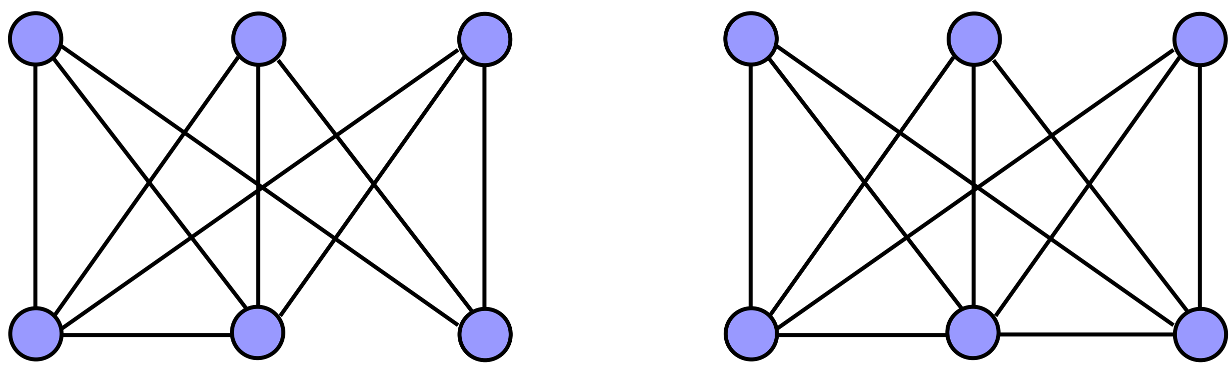
W_6

Tutte's Theorem: A graph G is 3-connected iff G is a wheel, or can be obtained from a wheel by a sequence of operations of type:

- The addition of a new edge.
- Replacing a vertex v of degree at least 4, by two adjacent vertices v_1 and v_2 such that:
  - each vertex formerly joined to v is connected to exactly one of v_1 and v_2.
  - Degrees of v_1 and v_2 are at least 3.

Example





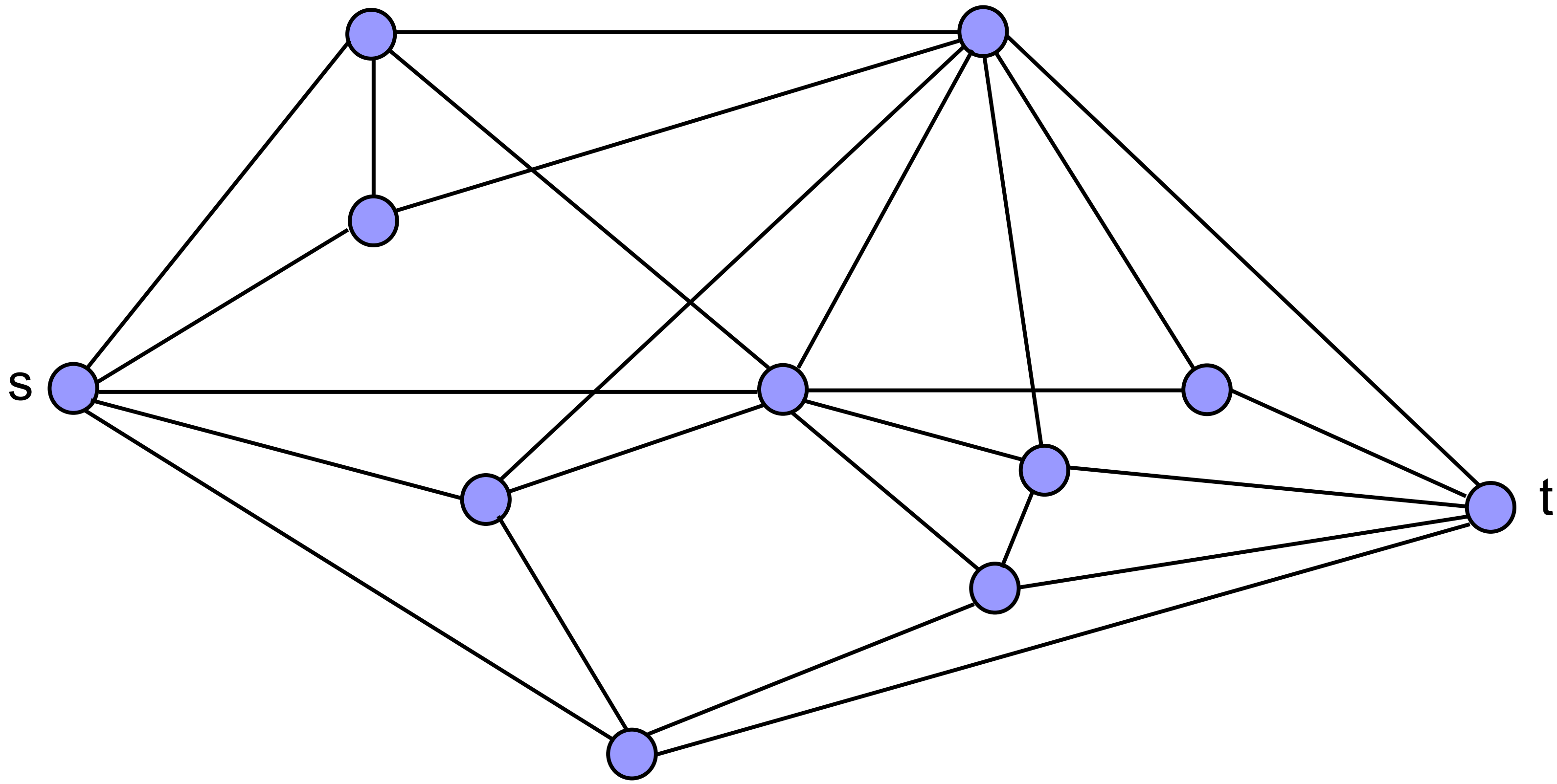
Menger's Theorem

- In 1927 Menger showed that:
  the connectivity of a graph is related to the number of disjoint paths joining distinct vertices in the graph.

Menger's Theorem: The minimum number of vertices separating two nonadjacent vertices s and t is the maximum number of disjoint s-t paths.

Whitney's Theorem (1932): A graph G is n-connected iff every pair of vertices of G are connected by at least n internally-disjoint (vertex-disjoint) paths.

Illustration



Variations of Menger's Theorem

- A analogous theorem to Menger's in which the pair of vertices are separated by a set of edges was discovered much later.

Theorem: For any two vertices of a graph, the maximum number of edge-disjoint paths connecting them, is equal to the minimum number of edges which disconnect them.

- Similarly, we can form the edge-form of Whitney's result:

Theorem: A graph G is n-edge-connected iff every pair of vertices of G are connected by at least n edge-disjoint paths.

Variations of Menger's Theorem – 2

Theorem:

For any two disjoint nonempty sets of vertices V_1 and V_2, the maximum number of disjoint paths connecting them, is equal to the minimum number of vertices which separate V_1 and V_2.

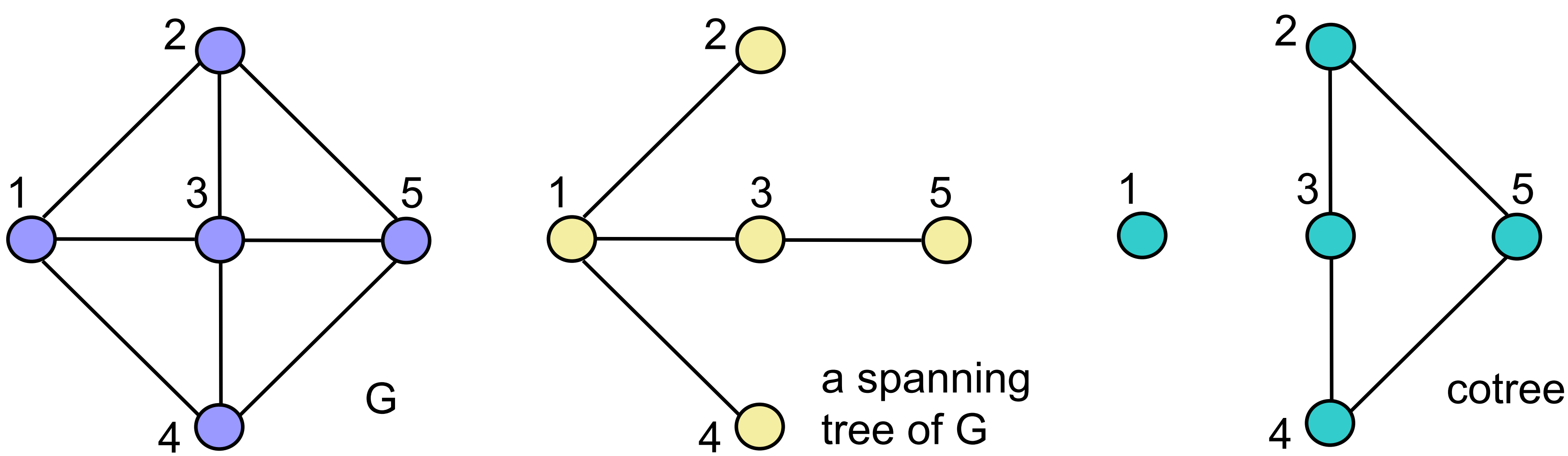
- No vertex of V_1 is adjacent to any vertex of V_2.

- All of the variations have corresponding digraph forms.
  - directed, undirected
  - specific vertices, general vertices, two sets of vertices
  - vertex-disjoint, edge-disjoint

A total of 2x3x2 = 12 theorems!

Circuits

- A cotree of a graph G w.r.t. a spanning tree T(V,E'): The set of edges E - E'.
  - If G has n vertices, then any cotree has $|E| - (n-1)$ edges.
- Any edge of a cotree is called a chord.



Ring-sum Operation

- Ring-sum $G_1 \oplus G_2$ of two graphs $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$, is the graph:

$$G_1 \oplus G_2 = (\,(V_1 \cup V_2),\ ((E_1 \cup E_2) - (E_1 \cap E_2))\,)$$

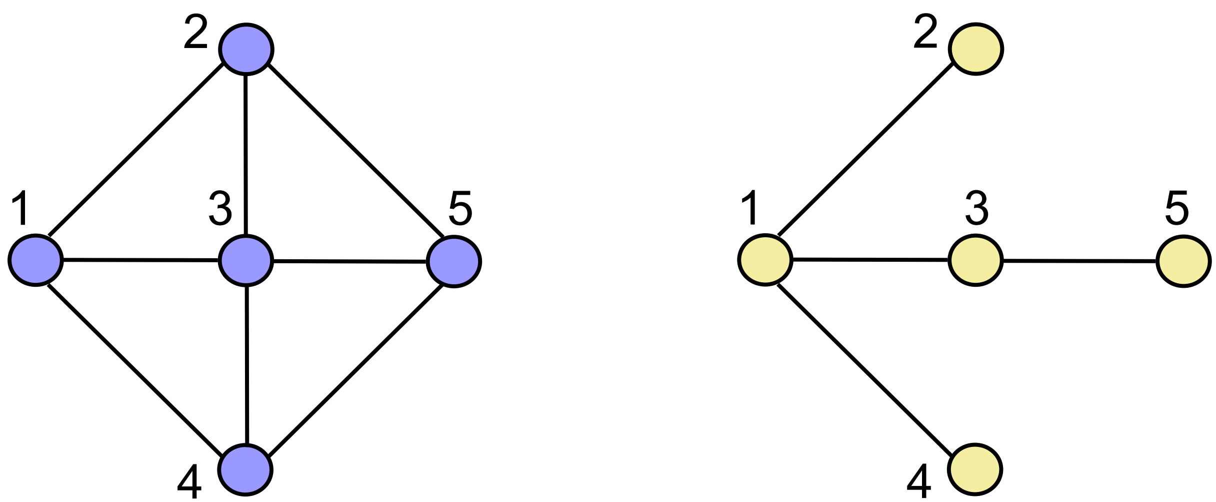
- Edges of a ring-sum consist of edges:
  - which are either in G_1 or G_2, but
  - which are not in both graphs.
- Ring-sum is both commutative and associative.

Fundamental Circuits

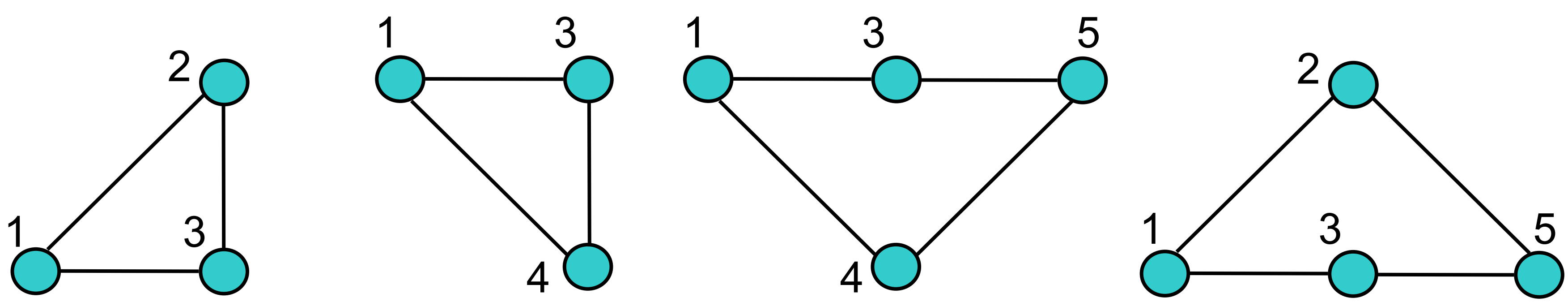
- The addition of a chord to a spanning tree creates precisely one circuit.
- The collection of these circuits w.r.t. a particular spanning tree is a set of fundamental circuits.
- Any arbitrary circuit of the graph may be expressed as a linear combination of the fundamental circuits using the operation ring-sum.

  The fundamental circuits form a basis for the circuit space.

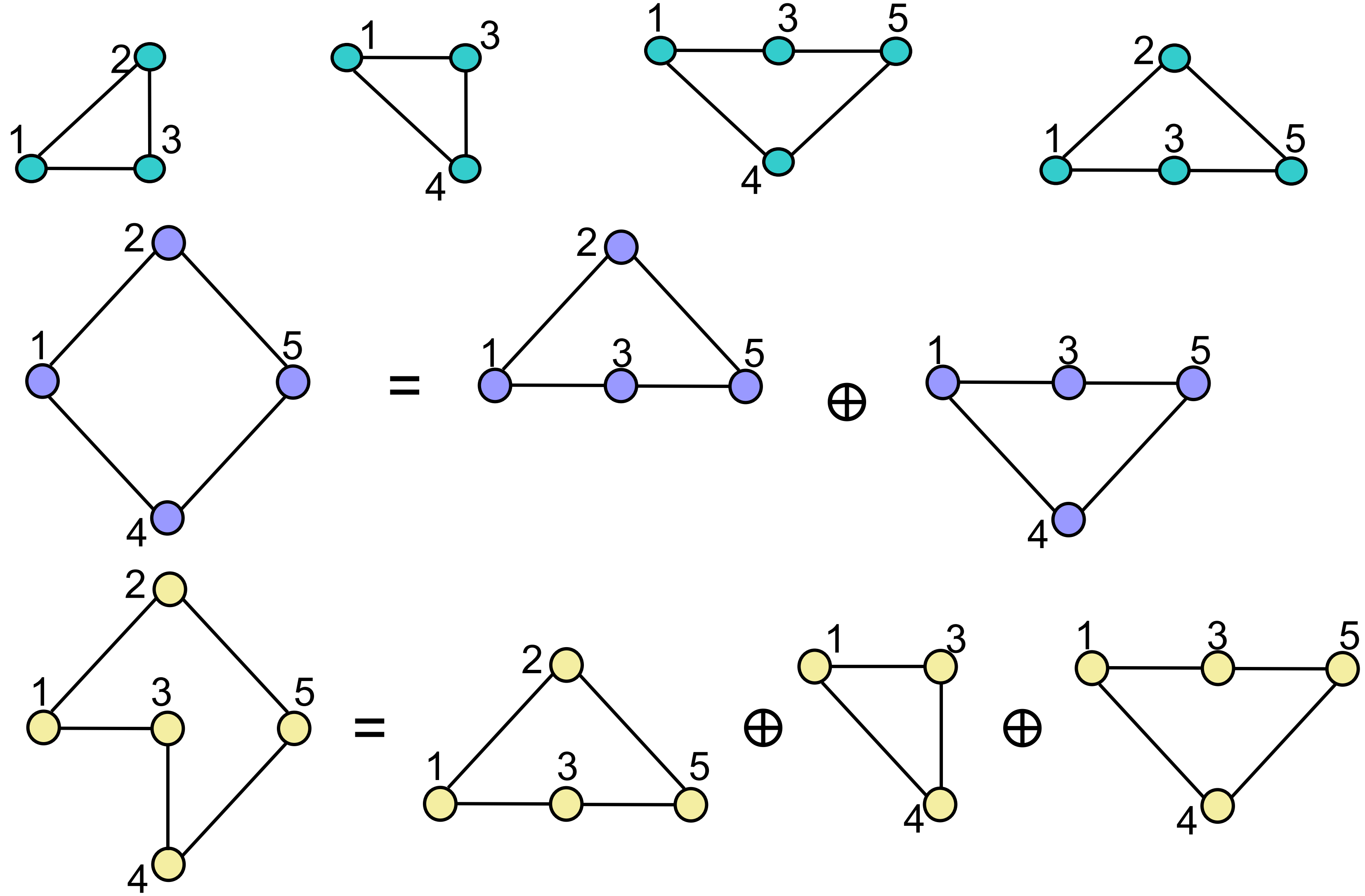
Fundamental Circuits Example



The fundamental set of circuits:



Some circuits of G expressed with fundamental circuits



Fundamental Circuit Theorems

Theorem: A set of fundamental circuits, w.r.t. some spanning tree of a graph G, forms a basis for the circuit space of G.

Corollary: The circuit space for a graph with $|E|$ edges and n vertices has dimension $(|E|-n+1)$.

Finding fundamental circuits

- Fundamental circuit set (FCS) can be found in polynomial-time.

```
Find a spanning tree T of G;
Find the corresponding cotree CT;
FCS = {};
for all ei=(vi,vi') ∈ CT do
  find the path from vi to vi' in T;
  denote the path by Pi;
  Ci = Pi ∪ {ei}
  FCS = FCS ∪ Ci;
endfor
```

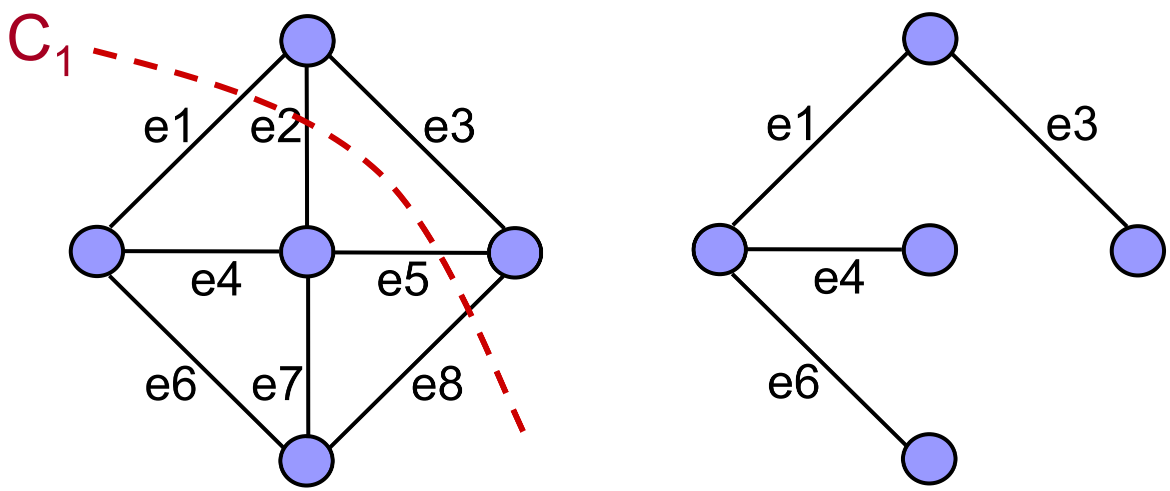
Fundamental Cut-sets

- A cut-set of a connected graph, is a set of edges whose removal would disconnect the graph.
- No proper subset of a cut-set will cause disconnection.
- A cut-set is denoted by the partition of vertices that it induces:
  - $[P, \overline{P}]$, where
    - P is the subset of vertices in one component,
    - $\overline{P} = V - P$

Fundamental Cut-sets

- Let T be a spanning tree of a connected graph.
- Any edge of T defines a partition of vertices of G:
  - The removal of this edge disconnects T
- Then:
  - There is a corresponding cut-set of G producing the same partition.
- This partition contains:
  - One edge of T, and
  - A number of chords of T.
- Such a cut-set is called a fundamental cut-set.

Example



- The set of fundamental cut-sets w.r.t. to T:
  - $C_1 = \{e_1,e_2,e_5,e_8\}$
  - $C_2 = \{e_4,e_2,e_5,e_7\}$
  - $C_3 = \{e_6,e_7,e_8\}$
  - $C_4 = \{e_3,e_5,e_8\}$

Fundamental Cut-set Theorems

Theorem: The fundamental cut-set w.r.t. some spanning tree of a graph G, forms a basis for the cut-sets of the graph.

Corollary: The cut-set space for a graph with n vertices has dimension $n - 1$.

Example

- Fundamental cut-sets:
  - $C_1 = \{e_1, e_2, e_5, e_8\}$
  - $C_2 = \{e_4, e_2, e_5, e_7\}$
  - $C_3 = \{e_6, e_7, e_8\}$
  - $C_4 = \{e_3, e_5, e_8\}$
- Some other cut-sets:
  1. $\{e_3, e_5, e_6, e_7\} = C_3 \oplus C_4$
  2. $\{e_1, e_4, e_6\} = C_1 \oplus C_2 \oplus C_3$
  3. $\{e_1, e_2, e_3\} = C_1 \oplus C_4$

Application: Constructing a Reliable Network

- Graph: representing a communication network

- Connectivity (*or edge-connectivity*):
  Smallest number of communication stations (*or communication links*) whose breakdown would jeopardize the communication.

- Higher the connectivity
  ⇒ the more reliable the network.

Application

- How do we create a reliable network, given the edge weights and nodes of the network?
  - Similar to connector problem
- Minimum spanning tree connects all nodes, and has minimum weight.
  - But, a tree is not very reliable!
- Generalization:
  Determine a minimum-weight k-connected spanning subgraph of a graph G.
  - G can be a complete graph or not.
  - k = 1: minimum spanning tree problem.

Application

- For values of k > 1, the problem is unsolved, and known to be difficult.
- However, the problem has a simple solution if:
  - G is a complete graph,
  - Each edge of G is assigned unit weight

Observation: For a complete graph of n vertices with unit edge weights, a minimum-weight k-connected spanning subgraph is:

- a k-connected graph on n vertices with as few edges as possible.

Application

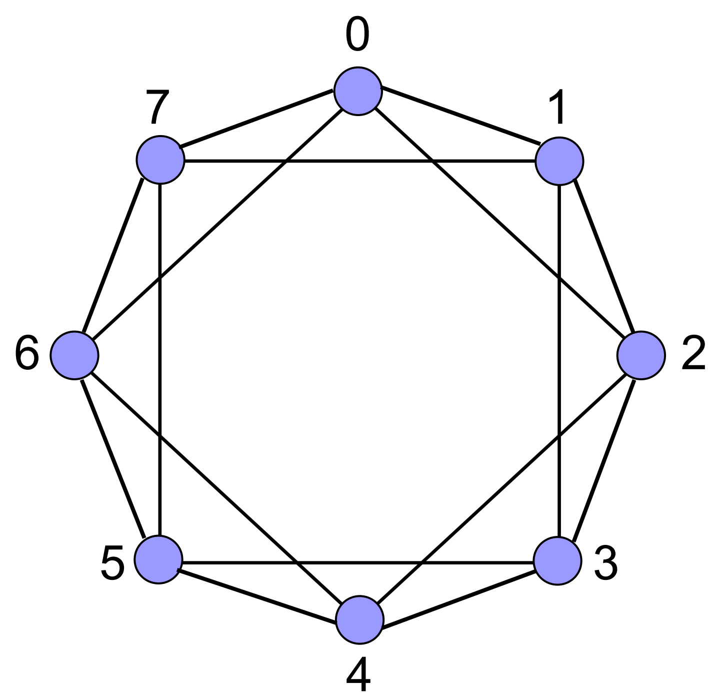
***f(m,n)*:** the least number of edges that an m-connected graph on n vertices can have (m < n).

$$f(m,n) \geq \{mn/2\}$$

- We will construct m-connected graphs $H_{m,n}$
- The structure of $H_{m,n}$ depends on the parities of m and n.

Case 1

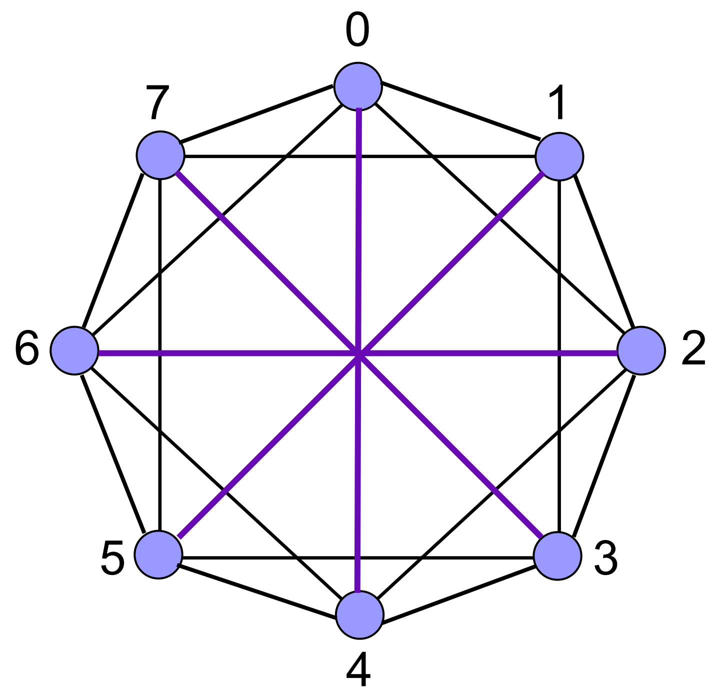
- m is even.
- Let $m = 2r$.
- Then, $H_{2r,n}$ is constructed as follows:
  - Vertices are numbered: 0, 1, 2, …, $n - 1$
  - Two vertices i, and j are joined if:
    $i - r \leq j \leq i + r$
    (addition in modulo)



$H_{4,8}$

Case 2

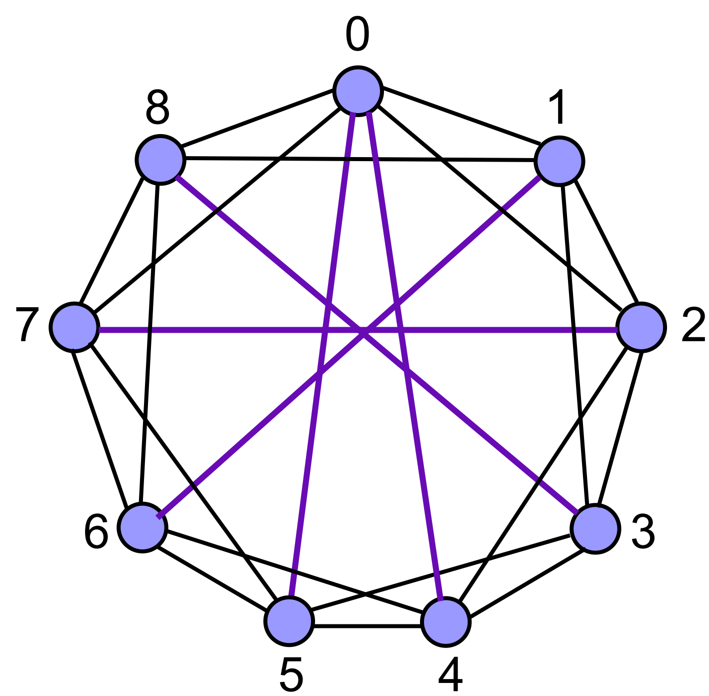
- m is odd, n is even.
- Let $m = 2r + 1$.
- Then, $H_{2r+1,n}$ is constructed as follows:
  - Draw $H_{2r,n}$
  - Add edges joining vertex i to vertex $i+n/2$ for: $1 \leq i \leq n/2$



$H_{5,8}$

Case 3

- m is odd, n is odd.
- Let $m = 2r + 1$.
- Then, $H_{2r+1,n}$ is constructed as follows:
  - Draw $H_{2r,n}$
  - Add edges joining:
    - 0 to $n - 1 / 2$
    - 0 to $n + 1 / 2$
    - vertex i to vertex $i+(n+1) / 2$ for $1 \leq i \leq (n - 1) / 2$



$H_{5,9}$

Resources:

- Edge, vertex-connectivity: Bondy&Murty: Ch.3
- Menger's Theorem: Harary: Ch.5
- Fundamental circuits and cut-sets: Gibbons: Sec.2.2

GRAPH THEORY and APPLICATIONS

Partitions

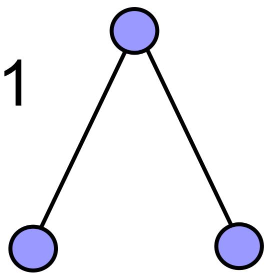
Degree Sequence

- The degrees $d_1, d_2, \ldots, d_v$ of the points of a graph form a sequence of nonnegative integers.
  - The sum of degree sequence is $2e$.
- Partition of a positive integer n: A list of unordered sequence of positive integers whose sum is n.
  - Example: n = 4
    4, 3 + 1, 2 + 2, 2 + 1 + 1, 1 + 1 + 1 + 1
- The degrees of a graph with no isolated vertices determine such a partition of $2e$.
  - To have a general definition for all graphs, we use an extended definition: instead of positive use nonnegative.

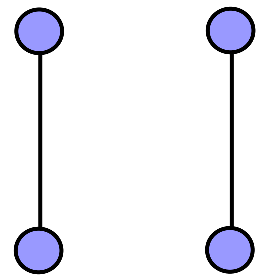
Partition of a graph

- The partition of a graph: Partition of $2e$ as the sum of the degrees of the points.
- Only two of the five partitions of 4 belong to a simple graph.

2+1+1



1+1+1+1



- A partition $\sum d_i$ of n into v parts is graphical if there is a graph G whose points have degrees d_i.

Two questions

- How can one tell whether a given partition is graphical?
- How can one construct a graph for a given graphical partition?
- An answer to the first question:
  by Erdös and Gallai (1960)
- Another answer to both:
  by Havel (1955) and by Hakimi (1962) (independently)

Havel and Hakimi's solution

Theorem: A partition $\Pi = (d_1, d_2, \ldots, d_v)$ of an even number into v parts with:

$$v - 1 \geq d_1 \geq d_2 \geq \ldots \geq d_v$$

is graphical if and only if the modified partition

$$\Pi' = (d_2 - 1, d_3 - 1, \ldots, d_{d1+1} - 1, d_{d1+2}, \ldots d_v)$$

is graphical.

Proof

- If Π' is graphical, then so is Π.
  - From a graph with partition Π' we can construct a graph with partition Π, by adding a new vertex adjacent to vertices of degrees:
    $$d_2 - 1, d_3 - 1, \ldots, d_{d1+1} - 1$$
- Let G be a graph with partition Π.
  - If a vertex of degree d_1 is adjacent to vertices of degrees d_i for i = 2 to d_1+1,
  - then, the removal of this vertex results in a graph with partition Π'.

Proof – 2

- Suppose that G has no such vertex.
- Assume v_1 is a vertex of degree d_1 for which:
  - the sum of the degrees of the adjacent vertices is maximum.
- Then:
  - there are vertices v_i and v_j with $d_i > d_j$
  - $v_1 v_j$ is an edge,
  - but $v_1 v_i$ is not.
- Therefore some vertex v_k is adjacent to v_i but not to v_j.
- Remove $v_1 v_j$ and $v_k v_i$. Add $v_1 v_i$ and $v_k v_i$. Repeat!

Constructing the graph

- The theorem gives an effective algorithm for constructing a graph with a given partition.

Corollary (Algorithm): A given partition
$\Pi = (d_1, d_2, \ldots, d_v)$ with:

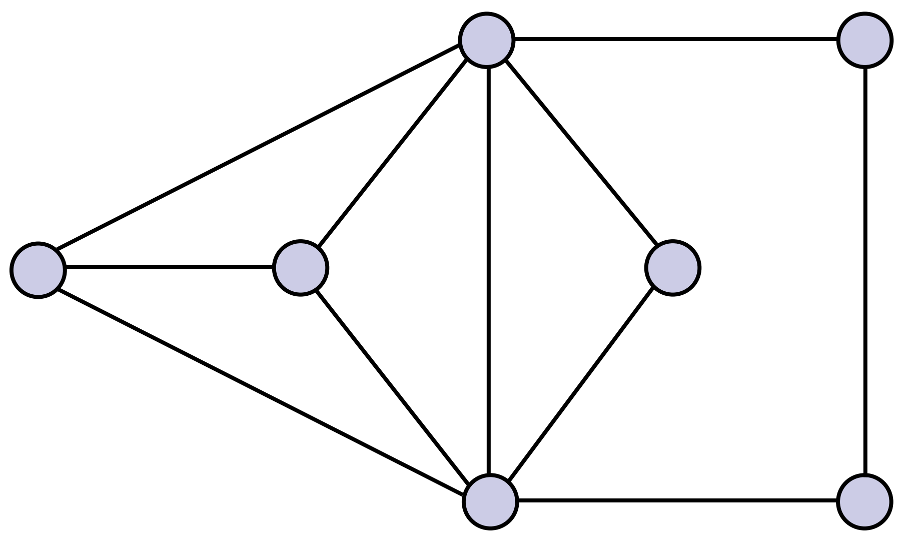
$$v-1 \geq d_1 \geq d_2 \geq \ldots \geq d_v$$

is graphical, if and only if the following procedure results in a partition with every term zero.

- Determine the modified partition Π' as in the theorem.
- Reorder the terms of Π' so that they are non-increasing, and call it partition Π_1.
- Go to step 1 and continue as long as non-negative terms are obtained.

Example

- $\Pi = (5,5,3,3,2,2,2)$
- $\Pi' = (4,2,2,1,1,2)$
- $\Pi_1 = (4,2,2,2,1,1)$
- $\Pi'_1 = (1,1,1,0,1)$
- $\Pi_2 = (1,1,1,1,0)$



The theorem of Erdös and Gallai

Theorem: Let $\Pi = (d_1, d_2, \ldots, d_v)$ be a partition of $2e$ into $v > 1$ parts.

$$d_1 \geq d_2 \geq \ldots \geq d_v$$

Then Π is graphical, if and only if, for each integer r, $1 \leq r \leq v-1$,

$$\sum_{i=1}^{r} d_i \leq r(r-1) + \sum_{i=r+1}^{v} \min(r, d_i)$$

- For a proof of this theorem, check Harary, p.59-61.