GRAPH THEORY and APPLICATIONS

Graph Coloring

Coloring

- Edge coloring: Coloring the edges of a graph, such that, no two adjacent edges are similarly colored.
  - A graph is k-edge-colorable if an edge-coloring using k colors exists.
  - Edge-chromatic index (number), $\psi_e(G)$: Minimum number of colors required for an edge-coloring of G.
- Vertex coloring: Coloring the vertices of a graph, such that, no two adjacent vertices are similarly colored.
  - A graph is k-vertex-colorable if a vertex-coloring using k colors exists.
  - Vertex-chromatic index (number), $\psi_v(G)$: Minimum number of colors required for a vertex-coloring of G.

Edge coloring

- An obvious lower bound: Maximum degree Δ of any vertex.
  - Edges meeting at any vertex must be differently colored.

Theorem: If G is a bipartite graph, then,

$$\psi_e(G) = \Delta.$$

Theorem: If G is a complete graph with n vertices, then:

$$\Psi_e(G) = \begin{cases} \Delta & \text{if n is even} \\ \Delta + 1 & \text{if n is odd} \end{cases}$$

Vizing's theorem

Theorem: For any simple graph G:

$$\Delta \leq \psi_e(G) \leq \Delta + 1$$

- For an arbitrary graph, the question of whether or not $\psi_e(G) = \Delta$, is NP-complete.
- A result applying to graphs without loops, due to Vizing:

$$\Delta \leq \psi_e(G) \leq \Delta + M$$

  - M (multiplicity): maximum number of edges joining any two vertices.
  - For any M, there exists a multi-graph such that $\psi_e(G) = \Delta + M$

Vertex coloring

- Vizing's theorem provides tight bounds on $\psi_e(G)$ for arbitrary graphs.
- Unfortunately, for $\psi_v(G)$, no theorem exists which gives such tight bounds based on simple criteria.
- There is no known polynomial-time algorithm to determine $\psi_v(G)$.
- For an arbitrary graph, the question of whether or not a graph contains a vertex coloring using less then k colors is NP-complete.

Vertex coloring

- An obvious bound:

Theorem: Any graph G is $(\Delta + 1)$-vertex colorable.

- The bound provided by the theorem can be far greater than the actual value of $\psi_v(G)$.
  - G may have a vertex arbitrarily large degree.
  - Ex: W_7

Theorem (Brooks): If G is not a complete graph, is connected, and has $\Delta \geq 3$, then G is Δ-vertex-colorable.

Simple heuristic

- Given that the problem of finding $\psi_v(G)$ does not have a polynomial time solution, it is necessary to think in terms of heuristics, and maybe approximation methods.
- Consider:

```
for i = 1 to n do
   while Ni[j] do j = j + 1;
   for all v ∈ A(i) do
      Nv[j] = true;
   endfor
   C(i) = j;
endfor
```

$N_i[j]$ = true
if a neighbor of i is colored in j

A(i): Adjacency list of i

C(i): Color of i

Simple heuristic

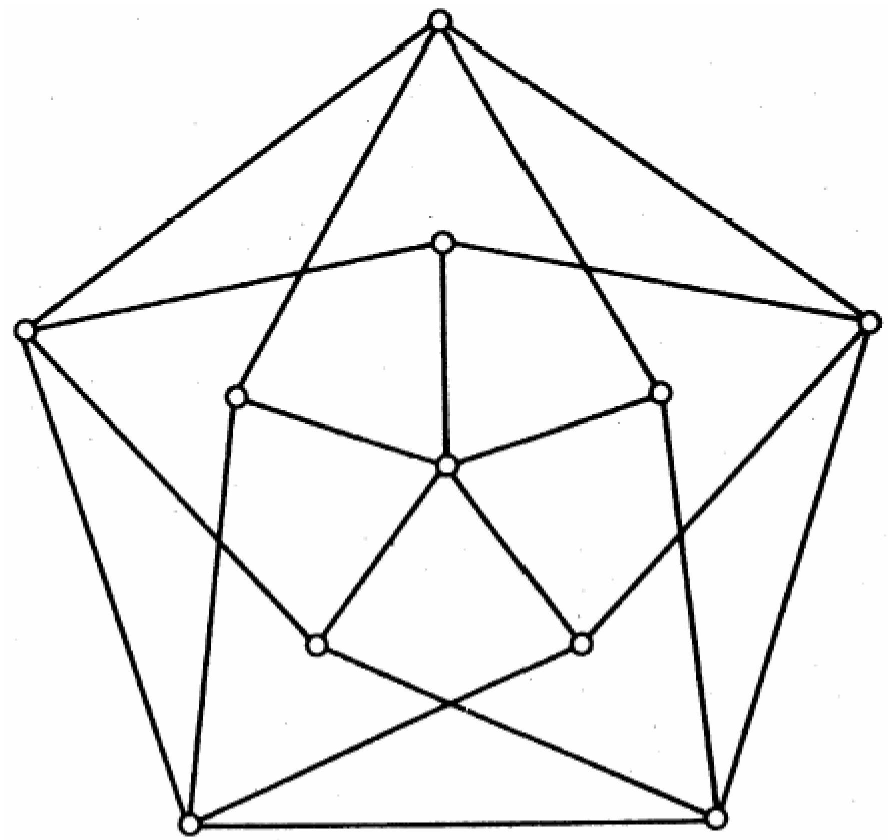
- The algorithm has $O(n^2)$-complexity.
- The behaviour of this algorithm is highly sensitive to the order in which the vertices are colored.
- There are no known polynomial-time algorithms for which the performance ratio is bound by a constant.
- The best performance ratio (due to Johnson): $O(n / \log(n))$.
- Garey & Johnson have shown that, if an approximation algorithm existed with a performance ratio of two or less, then it would be possible to find an optimal coloring in polynomial time.

An application

- Scheduling classes in an educational institution.
  - Acceptable teaching hours.
  - Many classes cannot be scheduled at the same time.
- Design the timetable, so that:
  - scheduled lectures are compressed into the shortest possible time.
- Solution:
  - Lectures: Vertices of the graph
  - Edges: connecting the vertices (lectures) which cannot be scheduled at the same time.
  - Color the vertices: Number of colors is the smallest time span within which the lectures can be scheduled.

Critical graphs

- A graph G is critical if $\psi_v(H) < \psi_v(G)$ for every proper subgraph H of G.
- A k-critical graph is one that is k-chromatic and critical.
- Every k-chromatic graph has a k-critical subgraph.



A 4-critical graph
(Grötzsch graph, 1958)

Critical graphs

Theorem: If G is k-critical, then $\delta \geq k - 1$.

(δ: minimum vertex degree)

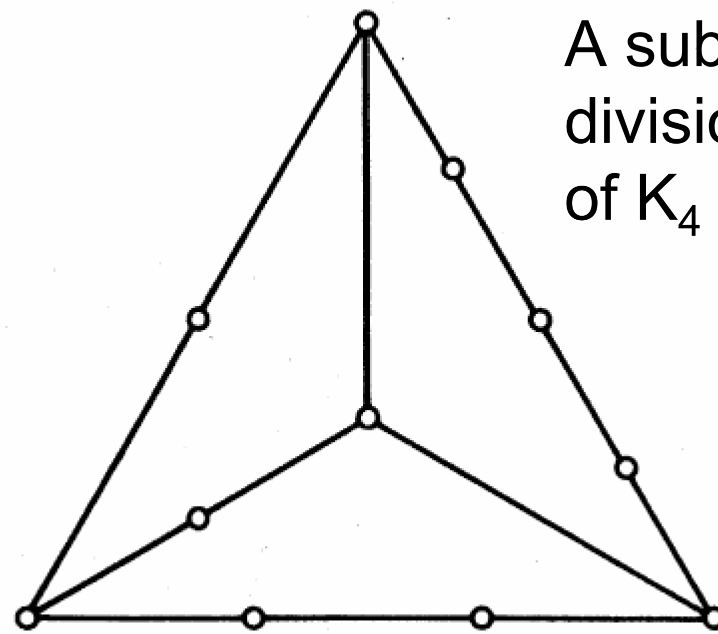
Corollary: Every k-chromatic graph has at least k vertices of degree at least $k - 1$.

Hajos' conjecture

- A subdivision of a graph G is a graph that can be obtained from G. by a sequence of edge subdivisions.
- A necessary condition for a graph to be k-chromatic, when $k \geq 3$:

Hajos' conjecture:

If G is k-chromatic, then G contains a subdivision of K_k.



A sub-division of K_4

Hajos' conjecture

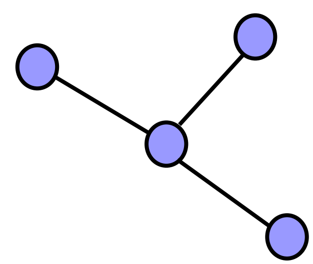
- Dirac settled the case k = 4:

Theorem: If G is 4-chromatic, then G contains a subdivision of K_4.

- Hajos' conjecture has not been settled in general case.
- It is known to be a very difficult problem.

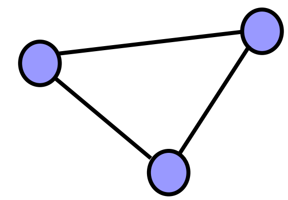
Chromatic polynomial

- Introduced by Birkhoff.
- $P_k(G)$: number of ways of vertex coloring the graph G with k colors.
  - A polynomial in k.
  - Referred to as chromatic polynomial of G.
- For the following graph:
  - Color the vertex of degree 3 first in k different ways,
  - Remaining vertices can each be colored in (k-1) ways.
- For any tree T with n vertices:
  $P_k(T_n) = k.(k-1)^{n-1}$



Chromatic polynomial

- Coloring the vertices of the graph on right:
  - Choice of k colors for the first vertex
  - k – 1 for the second
  - k – 2 for the third



- In general, for complete graphs:

$$P_k(K_n) = k! / (k-n)!$$

- For the graph Φ_n with n vertices and no edges:

$$P_k(\Phi_n) = k^n$$

- For k < ψ_v(G), chromatic polynomial equals 0.

Derivation of chromatic polynomial

- It is not easy to derive $P_k(G)$ for an arbitrary graph.
- The following theorem provides a systematic derivation:

Theorem: Let u and v be adjacent vertices in graph G. Then,

$$P_k(G) = P_k(G - (u,v)) - P_k(G \circ (u,v))$$

- G – (u,v) is derived by deleting edge (u,v)
- G ○ (u,v) is obtained by contracting the edge (u,v)

Derivation of chromatic polynomial

- Repeated application of the recursion formula will express $P_k(G)$ as a linear combination of chromatic polynomials of graphs with no edges.

- The formula of the theorem may also be applied in the form:

  $$P_k(G) = P_k(G + (u,v)) + P_k((G + (u,v)) \circ (u,v))$$

- In the second form, recursive evaluations of the formula leads to a linear combination of chromatic polynomials of complete graphs.

Derivation

- If G has a large number of edges, then second form will derive $P_k(G)$ more quickly.
- Whenever more than one edges arise between two vertices, only one edge is retained.
- $\psi_v(G)$ is the smallest value of k for which $P_k(G) > 0$.
- It is unlikely that $P_k(G)$ can be found in polynomial time.
  - This would imply that an efficient determination of $\psi_v(G)$ existed.

Clique and coloring

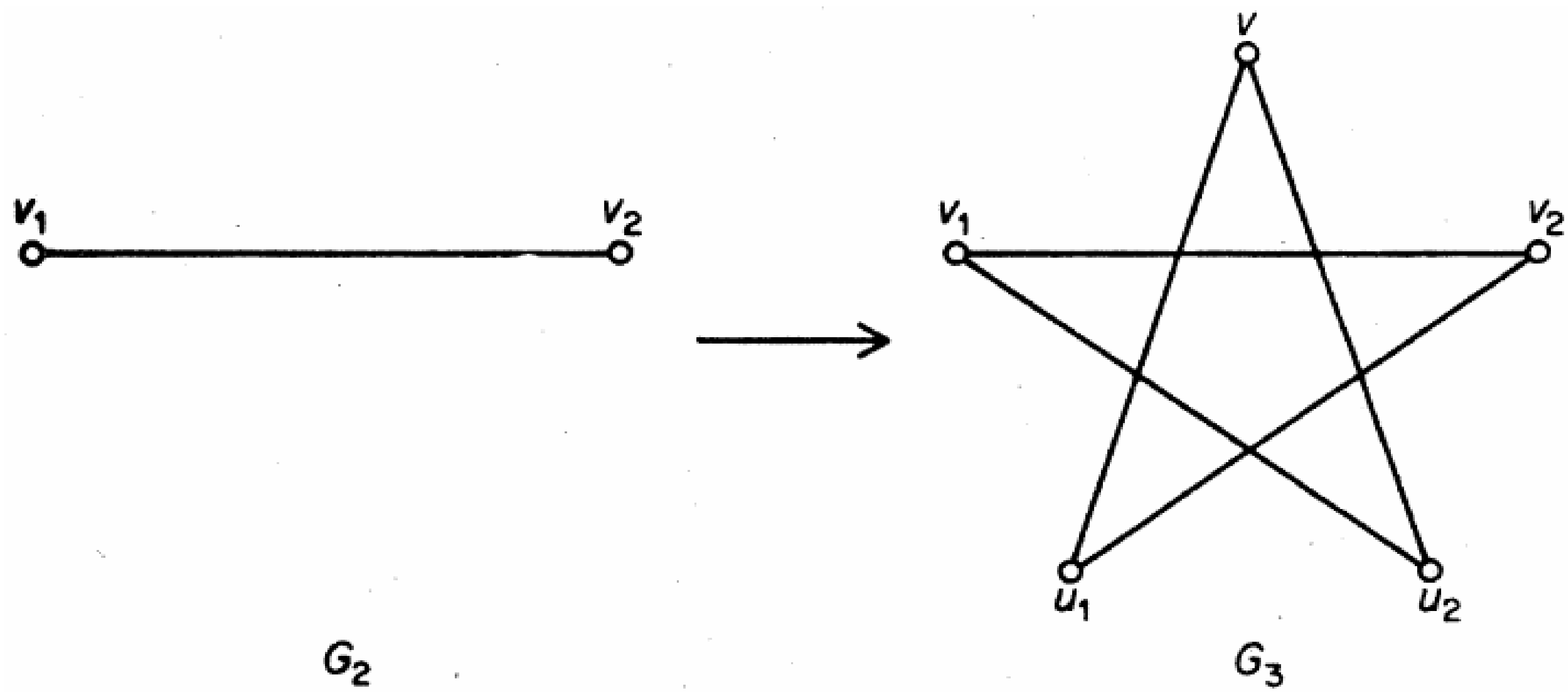
- In any vertex-coloring of a graph, the vertices in a clique must be assigned different colors.
- A graph with a large clique, has a high chromatic number.
- This leads us to believe that, all graphs with large chromatic number have large cliques.
- Dirac: Is there a graph with no triangles but arbitrarily high chromatic number?
- A recursive construction for such graphs was first described by B. Descartes (1954).

Mycielski's construction

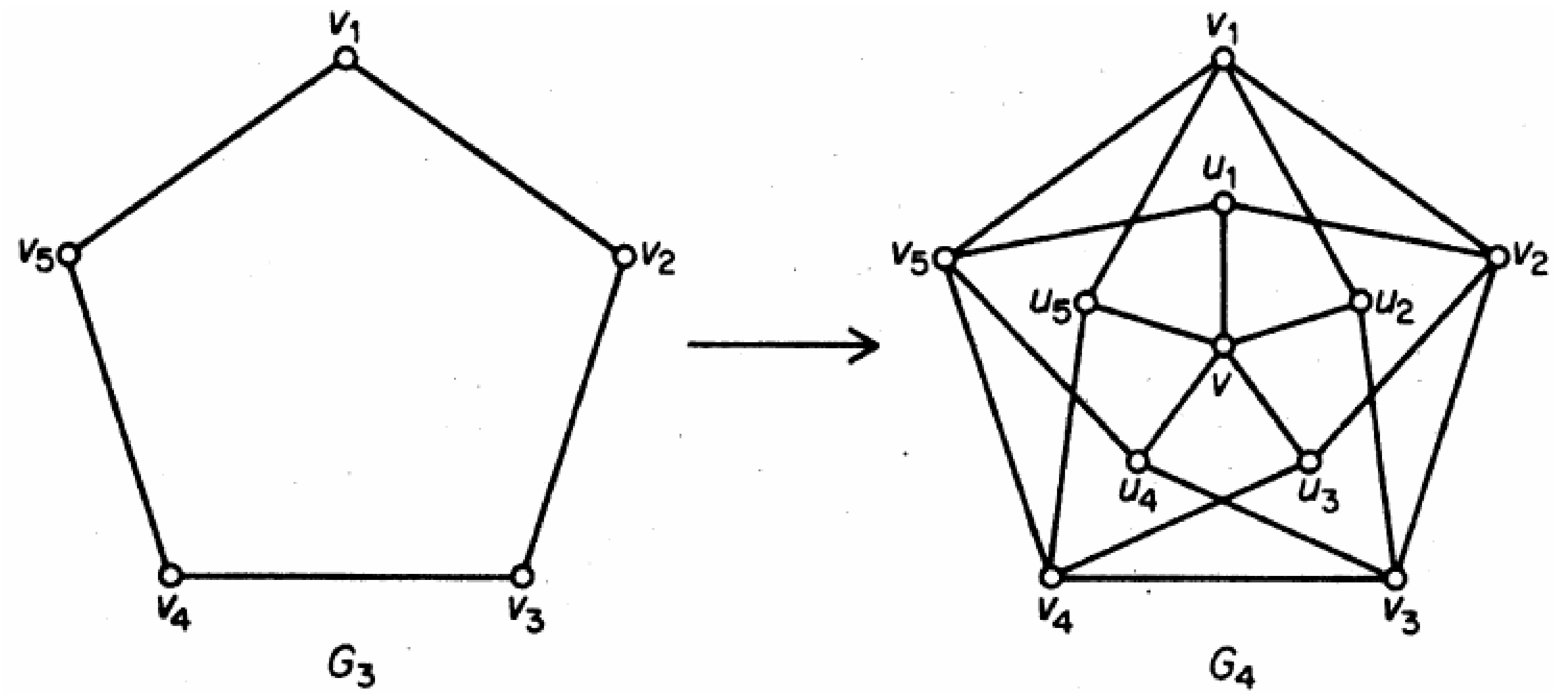
Theorem: For any positive integer k, there exists a k-chromatic graph containing no triangle.

- For k = 1 and k = 2, K_1 and K_2 have the required property.
- Suppose that we have already constructed a triangle-free graph G_k, with chromatic number $k \geq 2$
- Let the vertices of G_k be $v_1, v_2, \ldots, v_n$.
- Form a new graph G_{k+1} from G_k:
  - Add n+1 new vertices $u_1, u_2, \ldots, u_n, v$
  - for $1 <= i <= n$, join u_i to the neighbors of v_i and to v.

Mycielski's construction



Mycielski's construction



- This construction yields, for all k >= 2, a triangle-free k-chromatic graph on $3*2^{k-2} – 1$ vertices.

Relation to independent sets

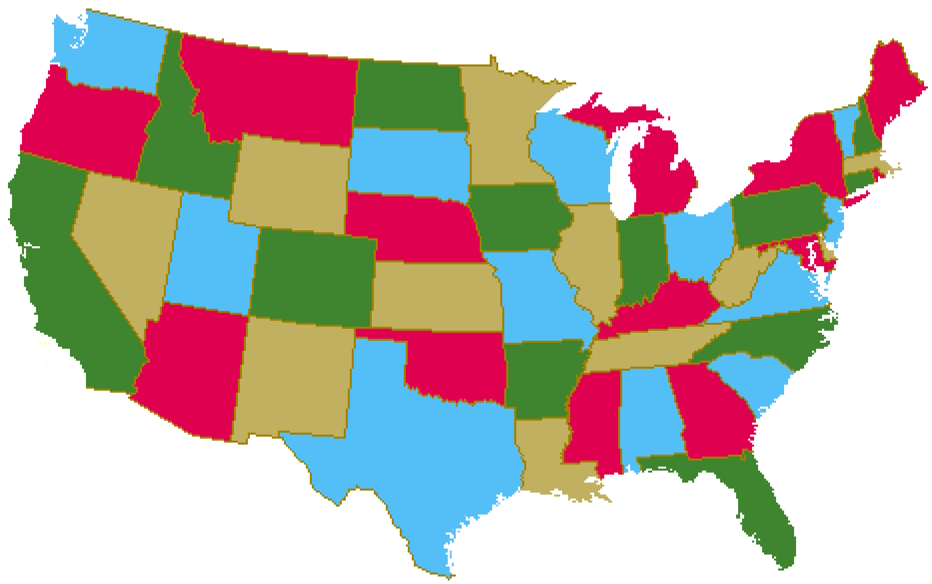
- A k-coloring of G where $(V_1, V_2, \ldots, V_k)$ is the partition, is canonical if:
  - V_1 is a maximal independent set of G
  - V_2 is a maximal independent set of $G - V_1$
  - V_3 is a maximal independent set of $G - (V_1 \cup V_2)$
  - and so on.
- If G is k-colorable, then there is a canonical k-coloring of G.

Face coloring

Four color theorem: Four colors are sufficient to color the regions of a planar map, so that bordering regions are differently colored.

(Region: face of a graph embedded in the plane.)

- This theorem was one of the best known unsolved problems, until 1976.



Face coloring

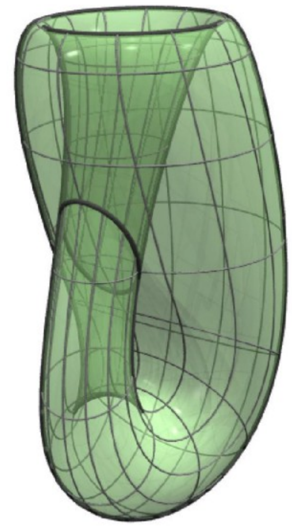
- For maps of genus $g \geq 1$, Heawood has shown that the following number of colors are sufficient:

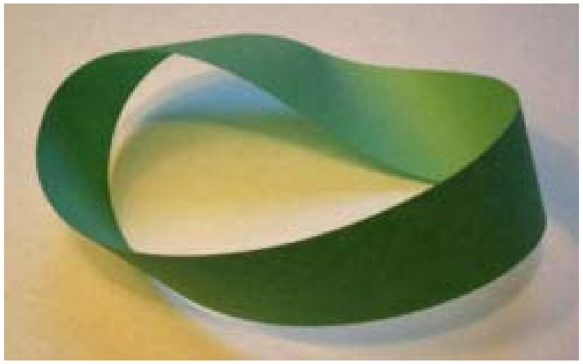
$$\left\lfloor \frac{7+\sqrt{(1+48g)}}{2} \right\rfloor$$

- Proof of this formula does not carry over for $g = 0$.
- The fact that is also necessary was proved by Ringel and Youngs (1968) with two exceptions:
  - the sphere (and plane), and
  - the Klein bottle.

Four color theorem

- When the four-color theorem was proved in 1976, the Klein bottle was left as the only exception.
  - For Klein bottle, the Heawood formula gives seven, but the correct bound is six.
  - The proof of four color conjecture dates back to 1840.
- The first mathematician to propose the four-color conjecture for the plane was Moebius.
- Many mathematician contributed to the current proof.





Proof

- Proof of four color theorem made massive use of computer time.
  - Period of trials and errors
  - Insight gained from the results and performances of computer programs.
- Would not have been achieved without the computer.
- A critic said: “A good mathematical proof is like a poem - this is a telephone directory!”
- Efforts still continue to achieve shorter, easier proofs.