GRAPH THEORY and APPLICATIONS

Factorization
Domination
Indepence
Clique

Factorization

Factor

- A factor of a graph G is a spanning subgraph of G, not necessarily connected.
- G is the sum of factors G_i, if:
  - G is the edge-disjoint union of G_i's.

  Such a union is called factorization.
- n-factor: A regular factor of degree n.
- If G is the sum of n-factors:
  - The union of n-factors is called n-factorization.
  - G is n-factorable.

1-factor

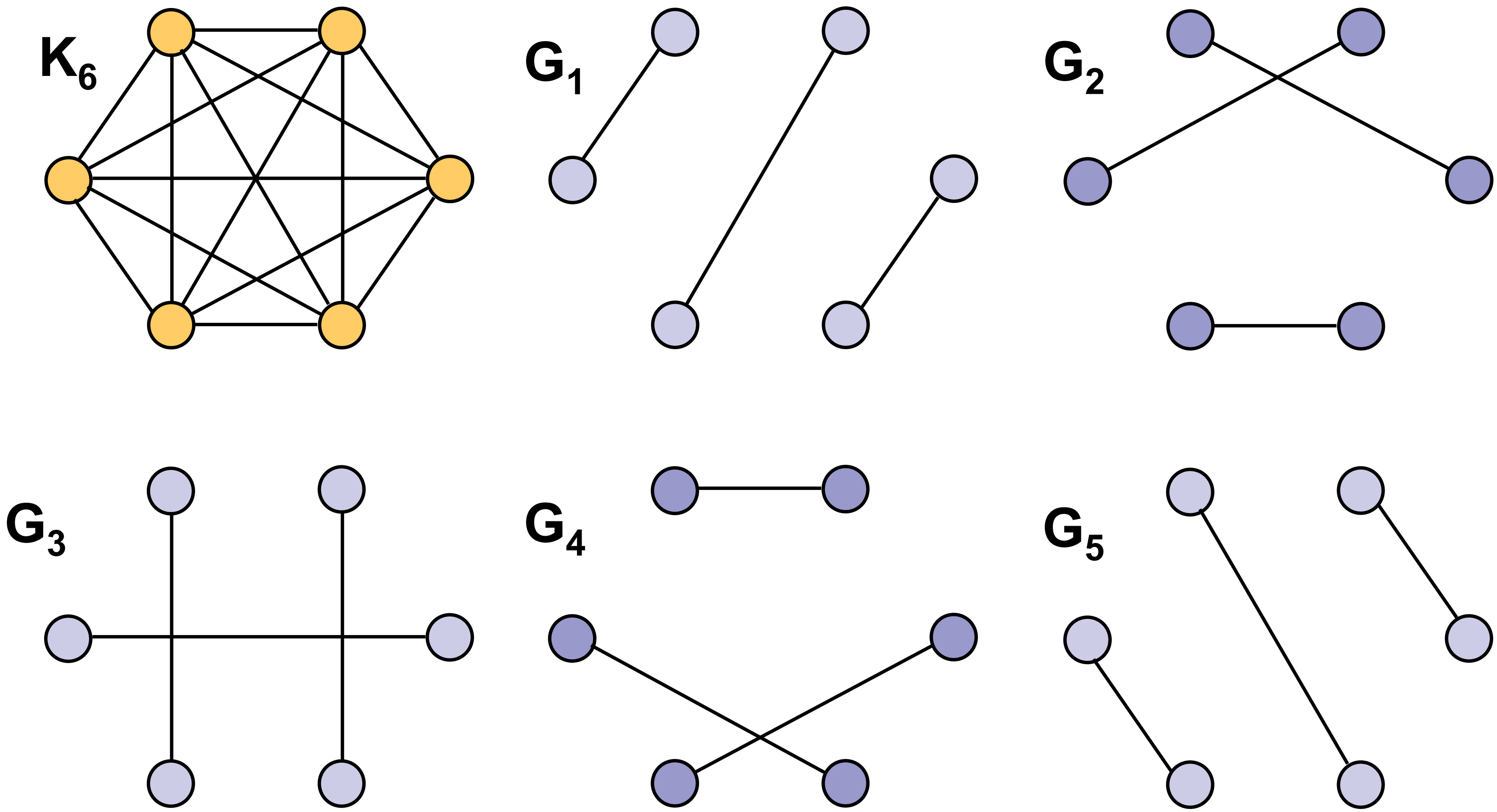
- When G has a 1-factor, G_1,
  - |V| is even.
  - The edges of G_1 are edge disjoint.
- K_{2n+1} cannot have a 1-factor, but K_{2n} can.

Theorem: The complete graph K_{2n} is 1-factorable.

We need to display a partition of the set E of edges of K_{2n} into (2n – 1) 1-factors.

- Denote the vertices: $v_1, v_2, \ldots, v_{2n}$
- Define for $\mathrm{i} = 1, 2, \ldots, 2n - 1$
  The sets $\mathrm{E_i} = \{v_i v_{2n}\} \cup \{v_{i\text{-}j} v_{i+j} \mid j = 1, 2, n-1\}$
  $i{+}1$ and $i - j$ are modulo$(2\mathrm{n} - 1)$ operations.

Example

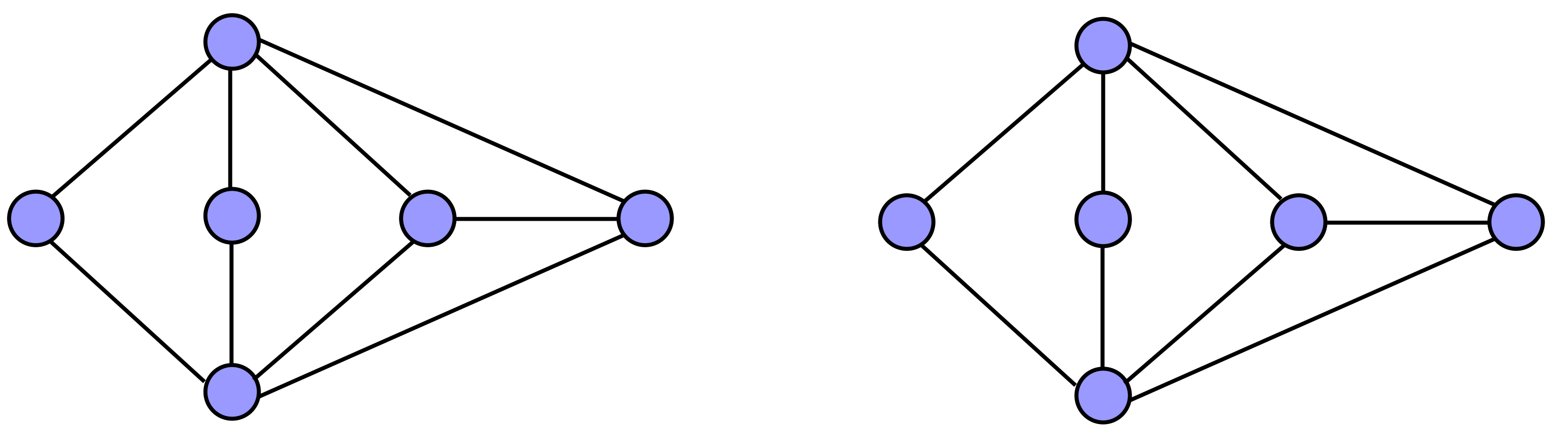


1-factors

- Complete bipartite graphs $K_{m,n}$ have no 1-factor if $n \neq m$.

Theorem: Every regular bipartite graph $K_{n,n}$ is 1-factorable.

Theorem: If a 2-connected graph has a 1-factor, then it has at least two different 1-factors.



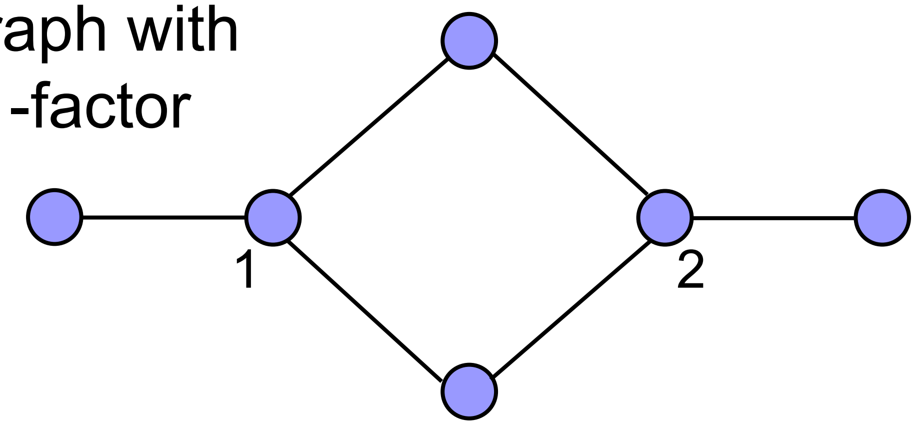
1-factor ≡ perfect matching

Tutte's Theorem: G(V,E) has a perfect matching (or a 1-factor) if and only if:

$$\Phi(G - V') \leq |V'| \quad \text{for all } V' \subset V$$

$\Phi(G - V')$: number of components of $(G - V')$ containing odd number of vertices.

A graph with no 1-factor



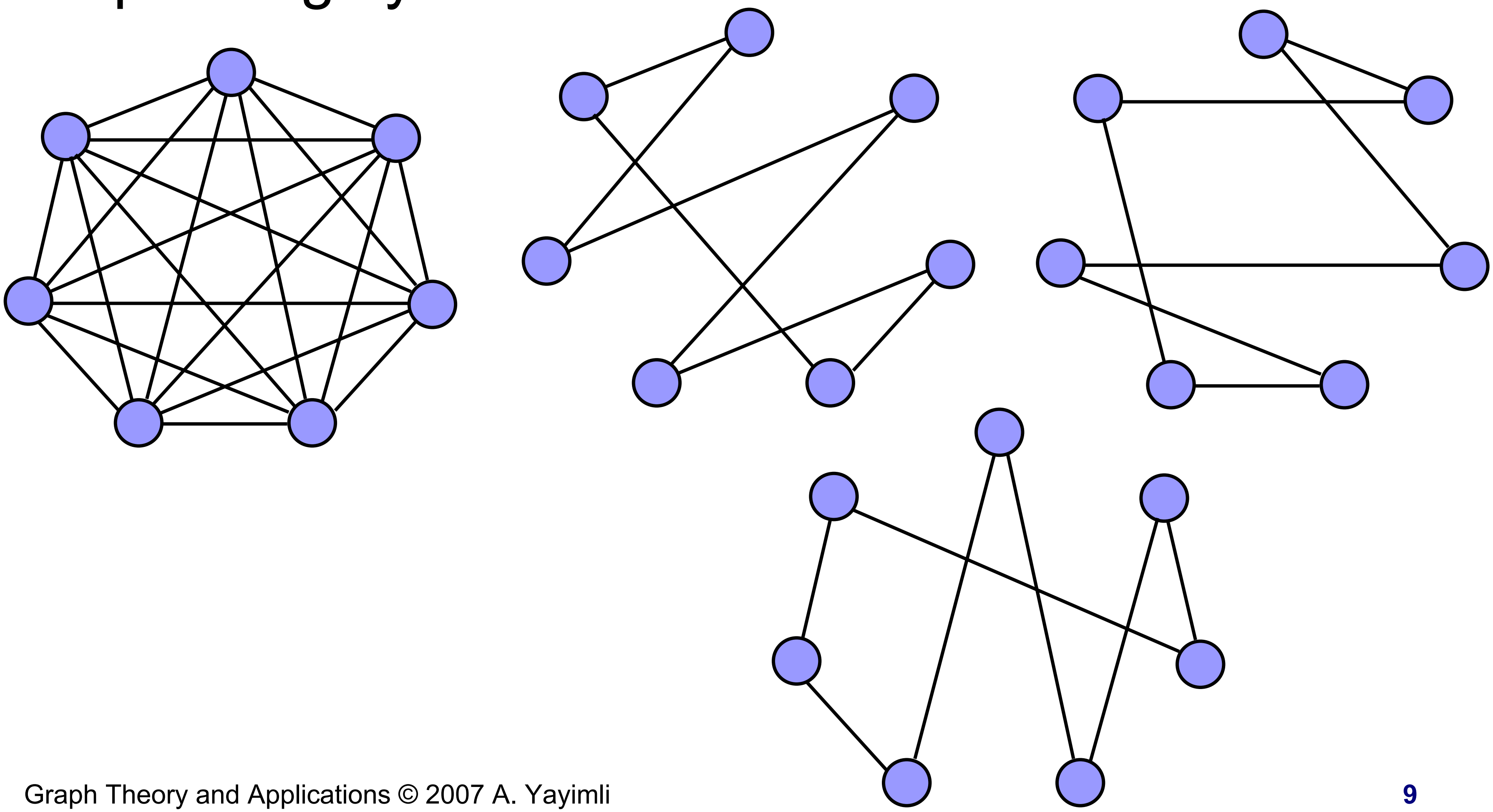
If vertex set S = {1,2} is removed: 4 components with odd number of vertices remain.

2-factorization

- If a graph is 2-factorable, then each factor is a union of disjoint cycles.
- If a 2-factor is connected, it is a spanning cycle.
- A 2-factorable graph must have all vertex degrees even.
- Complete graphs K_{2n} are not 2-factorable.
- K_{2n-1} complete graphs are 2-factorable.

2-factors

Theorem: The graph K_{2n+1} is the sum of n spanning cycles.



2-factors

Theorem: The complete graph K_{2n} is the sum of a 1-factor and n – 1 spanning cycles.

- If every component of a regular graph G of degree 2 is an even-length cycle, then G is also 1-factorable.
  - It can be represented as the sum of two 1-factors.

Theorem: Every bridgeless cubic graph is the sum of a 1-factor and a 2-factor.

- Example: Petersen graph.

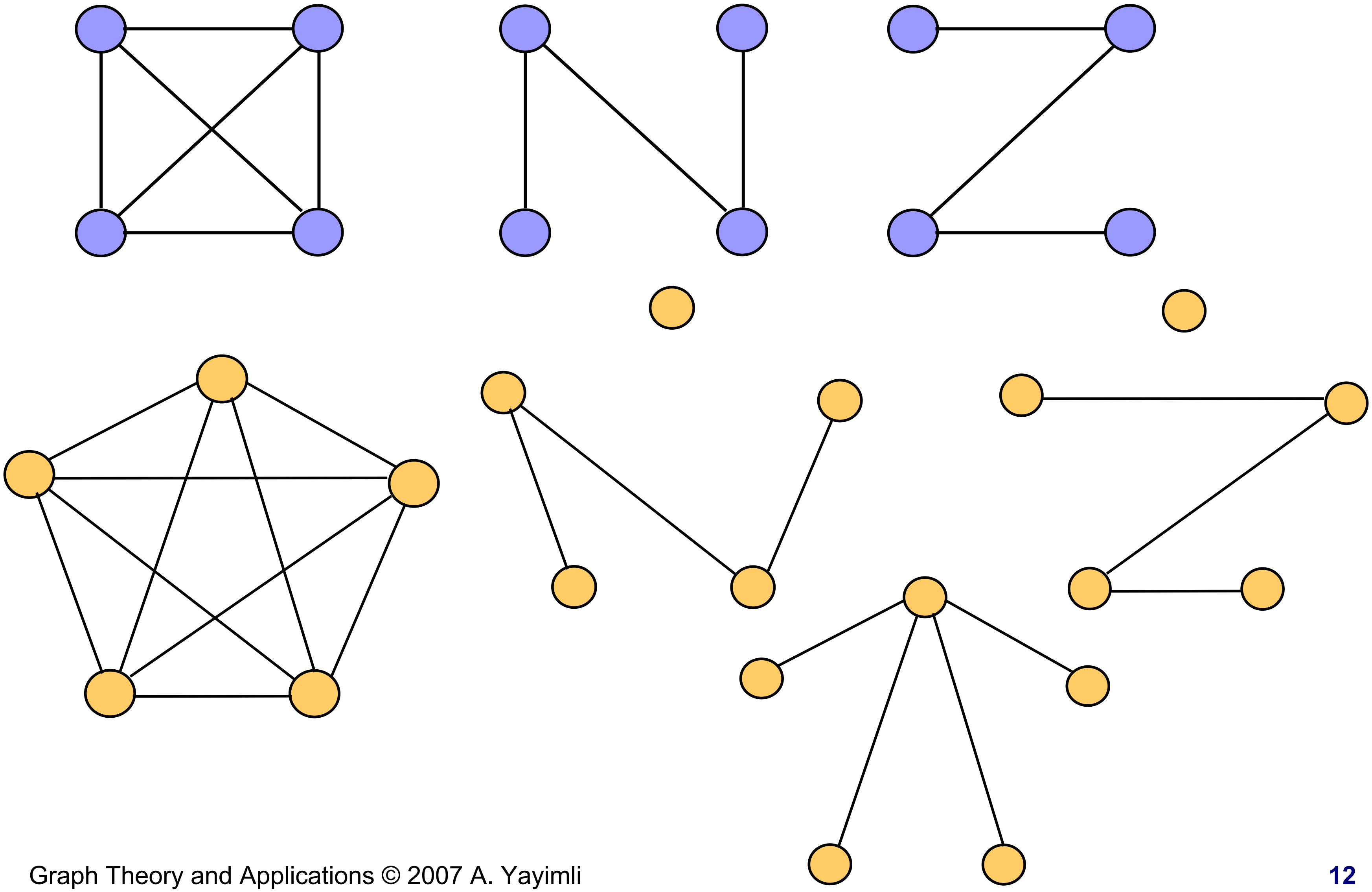
Arboricity

- Any graph G can be expressed as a sum of spanning forests
  - Let each factor contain only one egde.

Problem: Determine the minimum number of edge-disjoint spanning forests into which G can be decomposed.

- This number is arboricity of G, A(G).
- Example: $A(K_4) = 2$, $A(K_5) = 3$

Example



Arboricity

- A formula by Nash-Williams gives the arboricity of any graph.

Theorem: Let G be a non-trivial graph, and let:

- e_n be the maximum number of edges, in any subgraph of G having n vertices.

Then,

$$A(G) = \max_n \left\lceil \frac{e_n}{n-1} \right\rceil$$

Example: Fig. 9.8

Arboricity of complete graphs

Corollary:

- The arboricity of the complete graph K_n:

$$A(K_n) = \left\lceil \frac{e}{2} \right\rceil$$

- The arboricity of the complete bipartite graph $K_{m,n}$:

$$A(K_{m,n}) = \left\lceil \frac{m \cdot n}{m+n-1} \right\rceil$$

- The proof of Nash-Williams' formula does not gives a specific decomposition method.
- Beineke accopmlished the decomposition for complete graphs.

Decomposition of K_n

- For $n = 2m$, K_n can be decomposed into m spanning paths.
  - Label the vertices: $v_1, v_2, \ldots, v_{2m}$
  - Consider the n paths:

$$P_i = v_i v_{i-1} v_{i+1} v_{i-2} v_{i+2} \ldots v_{i+n-1} v_{i-n}$$

- For $n = 2m + 1$, the arboricity of K_n is $n+1$.
  - Take the same paths described.
  - Add an extra vertex labeled v_{2n+1} to each.
  - Construct a star, by joining v_{2n+1} to other $2n$ vertices.

Example: Fig.9.9

Dominating Set
Independence Set

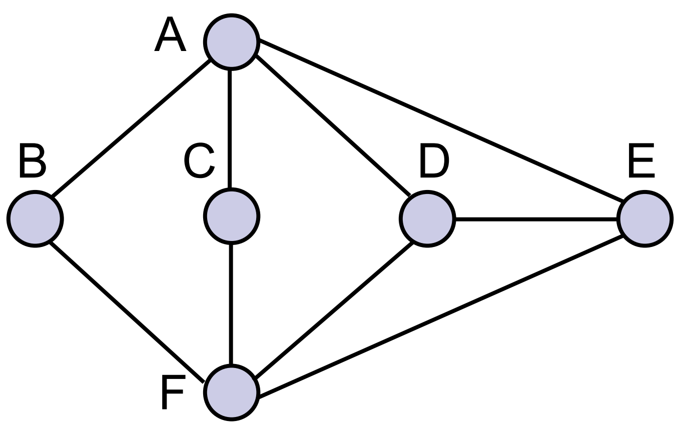
Domination-Independence

- Any vertex adjacent to a vertex v, is dominated by v.
- Any other vertex is independent of v.

- Independent Set: A subset of vertices of a graph where no two vertices are adjacent.
  - Maximal independent set: Any vertex not in the set is dominated by at least one vertex in it.
- Independence number: I(G), Cardinality of the largest independent set.

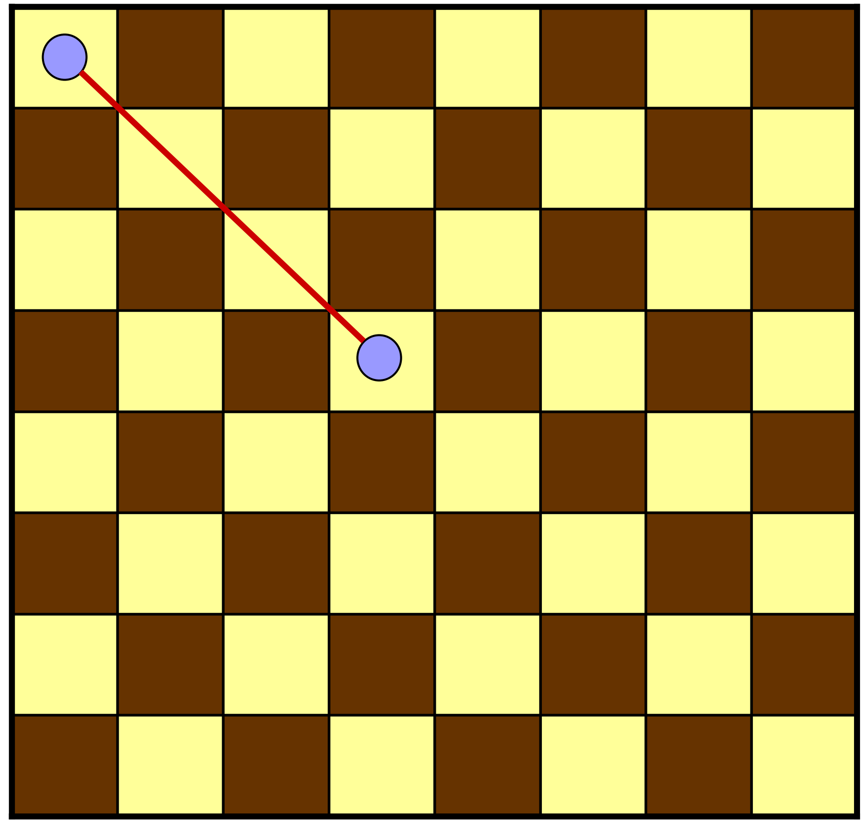
Domination–Independence

- Dominating Set: A subset of vertices of a graph where every vertex not in the subset is adjacent to at least one vertex in the subset.
  - Minimal dominating set: Contains no proper subset that is also a dominating set.
- Domination number: D(G), Cardinality of the smallest dominating set.



Example

- Board games usually provide illustrations of domination and independence.
- 8x8 chessboard: 64 vertices
  - An edge (u,v) implies that similar chess pieces placed at the squares u and v challenges one another.

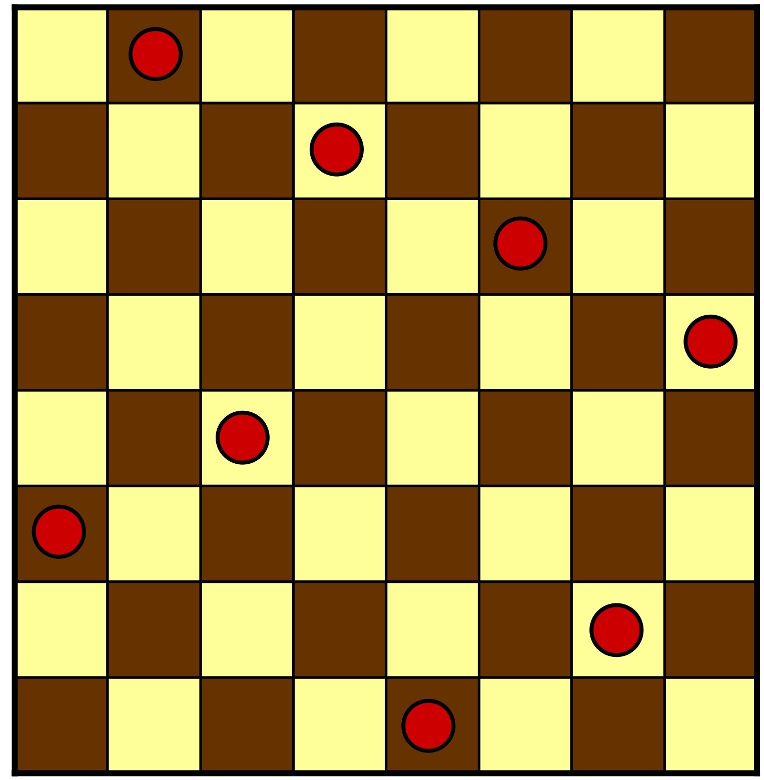


8 Queens

- Placing 8 queens on a chessboard so that:
  - no queen challenges another.

≡

- Finding a maximal independent set for the graph
  - containing the edges (u,v)
  - u and v: vertices corresponding the squares in the same row, column, or diagonal.



- There are 92 maximal independent sets
- I(G) = 8

Another queen problem

- What is the minimum number of queens
  - that can be placed on a standard chessboard
  - such that each square is dominated by at least one queen?

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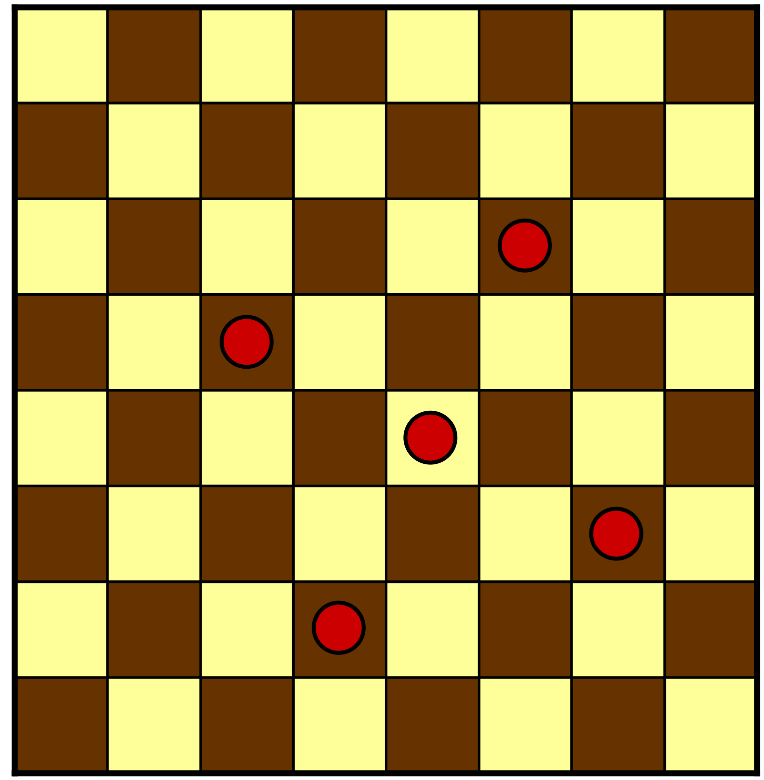
- Finding D(G) for the graph previously constructed.



D(G) = 5

A typical problem

Theorem: An independent set is also a dominating set if and only if it is maximal. Thus, I(G) $\geq$ D(G).

Consider the following problem:

- A community wishes to establish the smallest committee to represent a number of minority groups.
  - Any individual may belong to more than one group.
  - Every group has to be represented.

A problem

- The community = A graph
  - Vertices = individuals
  - Edges connect two individual in the same group
- Solution:
  - An independent set:
    No group should be represented more than once.
  - Which is also a dominating set:
    Each group must be represented.

- There no efficient algorithms to find I(G) or to find D(G) for an arbitrary graph G.

Finding Minimal Dominating Sets

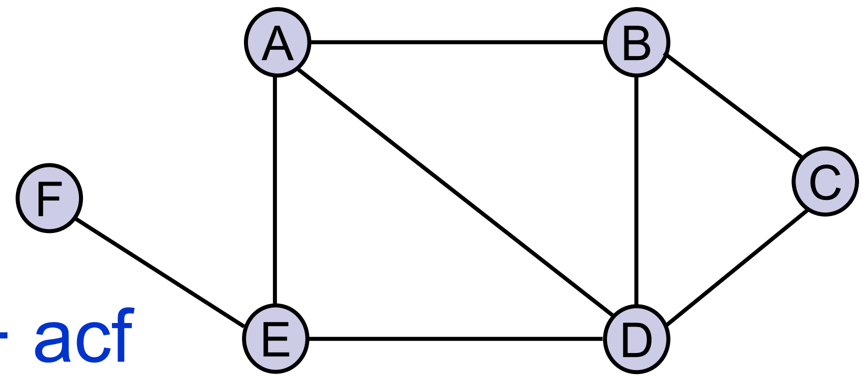
- A vertex v_i is dominated if:
  - v_i is in the dominating set, or
  - any of the vertices adjacent to v_i is in the dominating set.
- Then, we seek a minimal sum of products for the boolean expression:

$$A = \prod_{i=1}^{n} \left(v_i + v_i^1 + v_i^2 + \ldots + v_i^{d(v_i)} \right)$$

  - treating + as logical *or*, . as logical *and*.

Example

A = (a+b+d+e)(a+b+c+d)
(b+c+d)(a+b+c+d+e)
(a+d+e+f)(e+f)
= be + de + ce + bf + df + acf



- The six terms in the expression represent the minimal dominating sets:

  {b,e} {d,e} {c,e} {b,f} {d,f} {a,c,f}
- Five sets have cardinality of 2.
- D(G) = 2

Finding Maximal Independent Sets

- We enumerate the complement sets of maximal independent sets.
- For every edge (u,v), I^C must contain u or v or both.
- We must find the smallest sets I^C satisfying this condition for each edge.
- We obtain the minimum sum of products:

$$B = \prod_{(u,v)\in E} (u+v)$$

- Each term represents a set I^C, guaranteed to contain at least one endpoint from each edge.

Example

B = (a+b)(a+d)(a+e)(b+c)(b+d)(c+d)(d+e)(e+f)

= abce + abdf + acde +acdf +bde

Maximal independent sets:

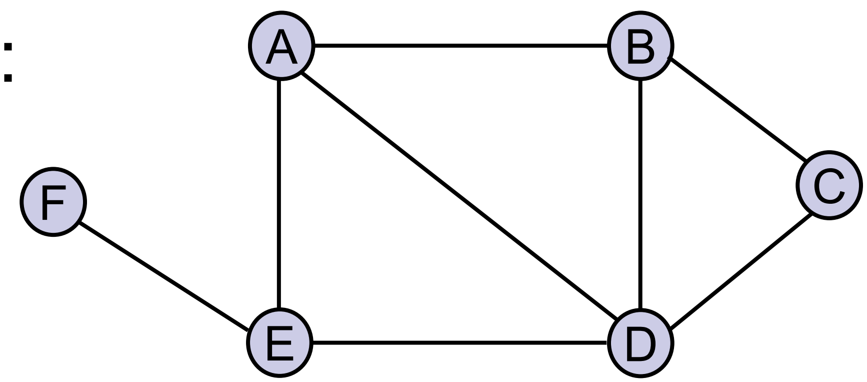
V – {a,b,c,e} = {d,f}

V – {a,b,d,f} = {c,e}

V – {a,c,d,e} = {b,f}

V – {a,c,d,f} = {b,e}

V – {b,d,e} = {a,c,f}   I(G) = 3



Clique

- Clique: Any subgraph of G, which is isomorphic to the complete graph K_i.
  - We can always partition the vertices of a graph into cliques.
- C(G): number of cliques in a partition which has the smallest possible number of cliques.

Theorem: For any graph G, $I(G) \leq C(G)$.

The presence or absence of large cliques is significant to the values of D(G) and I(G).

- All of the vertices in a clique are dominated by any one of its vertices.

- Determining whether an arbitrary graph contains a clique greater than a given size is an NP-complete problem.

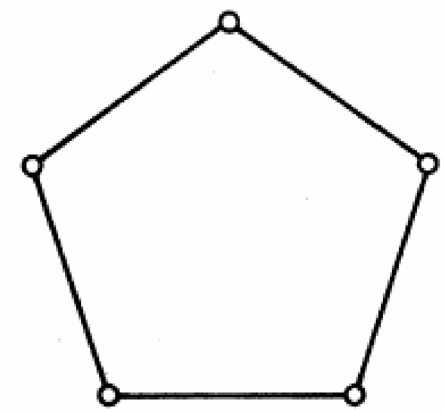
Ramsey Numbers

- Given any positive integer k and l, there exist a smallest integer r(k,l), such that:

  every graph on r(k,l) vertices contains:
  - a clique of k vertices, or,
  - an independent set of l vertices
- r(k,l) are called Ramsey numbers.
  - Example: r(3,3) = 6
- The determination of Ramsey numbers is a very difficult unsolved problem.
- Lower bounds are obtained by constructing suitable graphs.

Ramsey numbers

- 5-cycle contains no clique of size 3, nor an independent set of 3 vertices.



- **Theorem** (Erdös & Szekeres):
  For any two integers $k \geq 2$ and $l \geq 2$

  $r(k,l) \leq r(k, l - 1) + r(k - 1, l)$

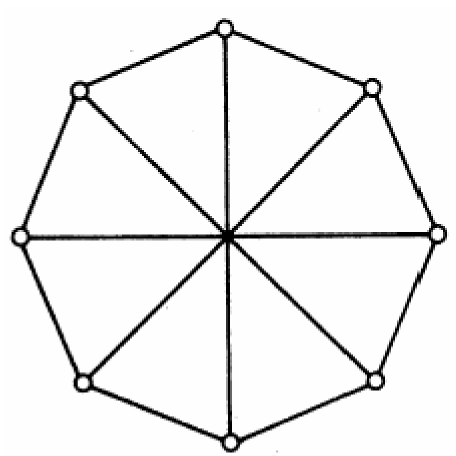
  Furthermore if $r(k, l - 1)$ and $r(k - 1, l)$ are both even, then strict inequality holds.

Ramsey Numbers known to date

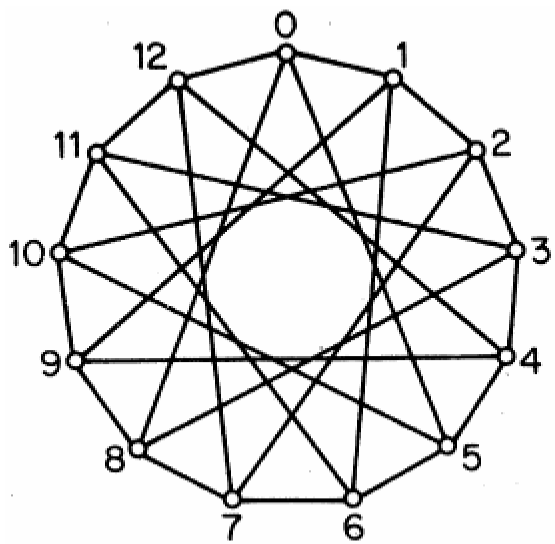
r,s	1	2	3	4	5	6	7	8	9	10
1	**1**									
2	**1**	**2**								
3	**1**	**3**	**6**							
4	**1**	**4**	**9**	**18**						
5	**1**	**5**	**14**	**25**[1995]	43–49					
6	**1**	**6**	**18**	35–41	58–87	102–165				
7	**1**	**7**	**23**	49–61	80–143	113–298	205–540			
8	**1**	**8**	**28**	56–84	101–216	127–495	216–1031	282–1870		
9	**1**	**9**	**36**	73–115	125–316	169–780	233–1713	317–3583	565–6588	
10	**1**	**10**	40–43	92–149	143–442	179–1171	289–2826	≤ 6090	580–12677	798–23556

Ramsey Graphs

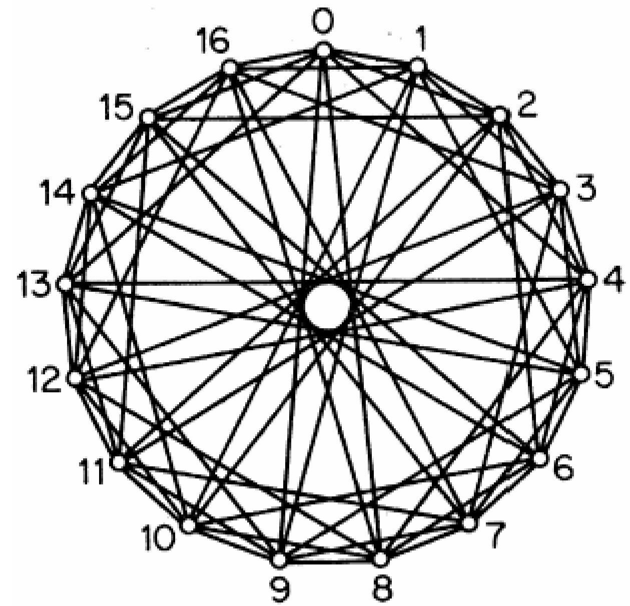
- A (k,l)-Ramsey graph is a graph:
  - on r(k,l) – 1 vertices
  - contains neither a clique of k vertices
  - nor an independent set of l vertices



A (3,4)-Ramsey graph



A (3,5)-Ramsey graph



A (4,4)-Ramsey graph

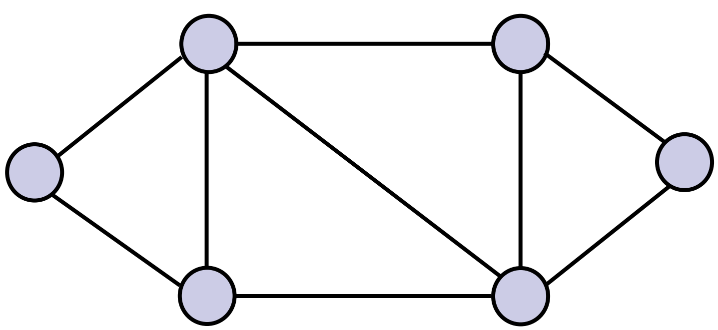
Size of a clique in a graph

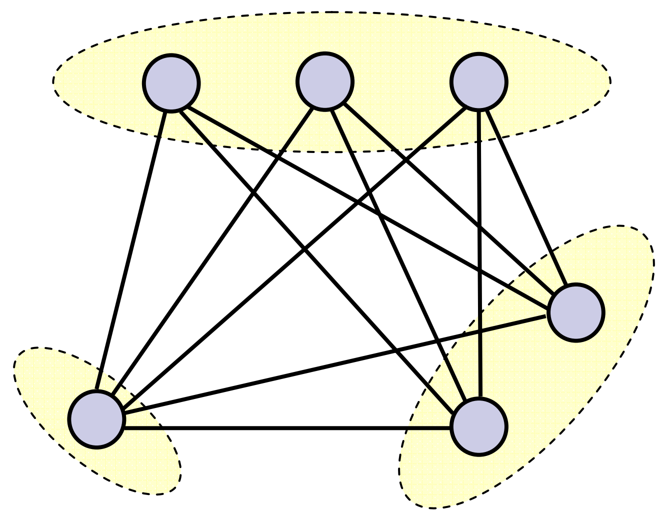
- Is there a limit to the number of edges that a graph may have, so that:
  - no subgraph is a clique of size k?
- Túran's theorem provides an upper bound.
- First, we need to revise another theorem by Erdös.

- Degree-majorized: A graph G is degree-majorized by another graph H if:
  - there is a one-to-one correspondence between the vertices of G and H
  - the degree of a vertex of H is greater than or equal to the degree of the corresponding vertex of G.

Theorem

Theorem (Erdös): If G is a simple graph, and does not contain a clique of size (i + 1), then,
G is degree-majorized by some complete i-partite graph P.
Also, if G has the same degree sequence as P, then, G is isomorphic to P.





Túran's Theorem

- $T_{j,n}$: j-partite graph with n vertices in which the parts are as equal in size as possible.

Theorem: If G is a simple graph which does not contain K_{j+1} then,

$$|E(G)| \le |E(T_{j,n})|$$

Also, only if G is isomorphic to $T_{j,n}$, then,

$$|E(G)| = |E(T_{j,n})|$$