On Relative Hamiltonian Diffeomorphisms

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Abstract. Let Ham(M,L) denote the group of Hamiltonian diffeomorphisms on a symplectic manifold M, leaving a Lagrangian submanifold $L \subset M$ invariant. In this paper, we show that Ham(M,L) has the fragmentation property, using relative versions of the techniques developed by Thurston and Banyaga.

1. Introduction

One of the main concerns in the study of automorphism groups of manifolds is whether the group is simple or perfect. The classical technique of Thurston [**7**] for showing that the group of C^∞ diffeomorphisms of a smooth manifold is simple (and hence perfect) requires two main properties of the group: fragmentation and transitivity. Banyaga[**1**] adaptes these techniques for the group of symplectomorphisms of a symplectic manifold. In this paper we prove that Ham(M,L): the group of relative Hamiltonian diffeomorphisms on a symplectic manifold M satisfies fragmentation property. Roughly the group consists of Hamiltonian diffeomorphisms on M that leave a fixed Lagrangian submanifold $L \subset M$ invariant, (see Ozan [**5**] for details). An automorphism group G on a manifold M is called 2-transitive if for all points in M x_1, x_2, y_1, y_2 with $x_1 \neq x_2$ and $y_1 \neq y_2$, there exists diffeomorphisms g_1, g_2 in G with $g_1(x_1) = y_1,\ g_2(x_2) = y_2$ and $supp(g_1) \cap supp(g_2) = \emptyset$. However Ham(M,L) is far from being transitive since it leaves a submanifold invariant. Indeed this is the main reason for the nonsimplicity of Ham(M,L). The main result of the paper is the Relative Fragmentation Theorem:

Theorem 1.1. *Let $\mathcal{U} = (U_j)_{j\in I}$ be an open cover of a compact, connected, symplectic manifold (M,ω) and h be an element of $\mathrm{Ham}(M,L)$ for a Lagrangian submanifold L of M. Then h can be written*

$$h = h_1 h_2 ... h_N,$$

where each $h_i \in \mathrm{Ham}_c(M,L),\ i = 1,..,N$ is supported in $U_{j(i)}$ for some $j(i) \in I$. Moreover, if M is compact, we may choose each h_i such that $R_{U_i, U_i\cap L}(h_i) = 0$, where we made the identification $U_{j(i)} := U_i$.

2010 *Mathematics Subject Classification.* Primary 53D05 53D12 53D35 57S05.
Key words and phrases. Hamiltonian diffeomorphisms, Lie group, Lagrangian submanifold.

The relative Calabi homomorphism $R_{U_i, U_i \cap L}$ will be defined in the next section. $\mathrm{Ham}_c(M, L)$ is the subgroup consisting of compactly supported elements.
For preliminaries and the techniques on the classical diffeomorphism groups Banyaga's book [**1**] can be consulted. In this work we followed his exposition of the forementioned techniques of Thurston and Banyaga.

2. Relative Hamiltonian Diffeomorphisms

Let (M^{2n}, ω) be a symplectic manifold, i.e. ω is a closed 2-form such that ω^n is a volume form on M. The group of symplectomorphisms is defined as

$$\mathrm{Symp}(M, \omega) = \{\phi \in \mathrm{Diff}^\infty(M) \mid \phi^*\omega = \omega\}.$$

$\mathrm{Symp}(M, \omega)$ is by definition equipped with C^∞-topology and as first observed by Weinstein in [**8**] it is locally path connected. Let $\mathrm{Symp}_0(M, \omega)$ denote the path component of $id_M \in \mathrm{Symp}(M, \omega)$. For any $\psi \in \mathrm{Symp}_0(M, \omega)$, let $\psi_t \in \mathrm{Symp}(M, \omega)$ for all $t \in [0, 1]$, such that $\psi_0 = id_M$ and $\psi_1 = \psi$. There exists a unique family of vector fields (corresponding to ψ_t)

$$X_t : M \longrightarrow TM \quad \text{such that} \quad \frac{d}{dt}\psi_t = X_t \circ \psi_t. \tag{1}$$

The vector fields X_t corresponding to ψ_t (the notation $\dot{\psi}_t$ is also used commonly to denote X_t) are called symplectic since they satisfy $\mathcal{L}_{X_t}\omega = 0$, where $\mathcal{L}_{X_t}\omega$ denotes the Lie derivative of the form ω along the vector field X_t. By Cartan's formula

$$\mathcal{L}_{X_t}\omega = i_{X_t}(d\omega) + d(i_{X_t}\omega).$$

Hence X_t is a symplectic vector field if and only if $i_{X_t}\omega$ is closed for all t. If moreover $i_{X_t}\omega$ is exact, that is to say $i_{X_t}\omega = dH_t$, $H_t : M \to \mathbb{R}$ a family of smooth functions, then X_t are called Hamiltonian vector fields. In this case the corresponding diffeomorphism ψ is called a Hamiltonian diffeomorphism and H_1 is a Hamiltonian for ψ. The Hamiltonian diffeomorphisms form a group as a subgroup in the identity component of the group of symplectomorphisms, $\mathrm{Ham}(M, \omega) \subseteq \mathrm{Symp}_0(M, \omega)$.

Let (M, ω) be a connected, closed symplectic manifold, L a Lagrangian submanifold of M. Denote by $\mathrm{Symp}_0(M, L)$ the identity component of the group of symplectomorphisms of M that leave L setwise invariant and by $\widetilde{\mathrm{Symp}_0}(M, L)$ its universal cover. Then the restriction of the flux homomorphism to $\widetilde{\mathrm{Symp}_0}(M, L)$ is a well defined homomorphism onto $H^1(M, L)$ (see [**5**]), given by

$$\mathrm{Flux}(\{\psi_t\}) = \int_0^1 [\mathrm{i}_{\mathrm{X_t}}\omega]\mathrm{dt}$$

where $\{\psi_t\} \in \widetilde{\mathrm{Symp}_0}(M, L, \omega)$ denotes the homotopy class of smooth paths $\psi_t \in \mathrm{Symp}_0(M, L)$ with fixed ends $\psi_0 = id$, $\psi_1 = \psi$ and X_t is the vector field defined by $\frac{d}{dt}\psi_t = X_t \circ \psi_t$. Note that since ψ_t leaves L invariant, for any $p \in L$, $X_t(p) \in T_pL$.
Notation: Let M be a manifold, $L \subset M$ a submanifold. If f is meant to be a map of M that leave L setwise invariant then we write $f : (M, L) \to (M, L)$.

Let $\mathrm{Ham}(M, L) \subset \mathrm{Symp}_0(M, L)$ be the subgroup consisting of symplectomorphisms ψ such that there is a Hamiltonian isotopy $\psi_t : (M, L) \to (M, L)$, $t \in [0, 1]$ with $\psi_0 = id$, $\psi_1 = \psi$; i.e. ψ_t is a Hamiltonian isotopy of M such that $\psi_t(L) = L$

for any $t \in [0,1]$. So if X_t is the vector field associated to ψ_t we have $i_{X_t}\omega = dH_t$ for $H_t : M \to \mathbb{R}$. Since L is Lagrangian ($w|_L = 0$), H_t is locally constant on L. We have the following characterization which is analogous to its absolute version (see Thm 10.12 in [**4**]).

Theorem 2.1. *([**5**]) $\psi \in \mathrm{Symp}_0(M,L)$ is a Hamiltonian symplectomorphism if and only if there exists a symplectic isotopy $\psi_t : [0,1] \to \mathrm{Symp}_0(M,L)$ such that $\psi_0 = id$, $\psi_1 = \psi$ and $\mathrm{Flux}(\{\psi_t\}) = 0$. Moreover, if $\mathrm{Flux}(\{\psi_t\}) = 0$ then $\{\psi_t\}$ is isotopic with fixed end points to a Hamiltonian isotopy through points in $\mathrm{Symp}_0(M,L)$.*

2.1. Relative Calabi Homomorphism. The relative version of the Calabi homomorphism is defined by the same formula of its absolute version. Let (M^{2n},ω) be a noncompact symplectic manifold and $L^n \subset M^{2n}$ be a Lagrangian submanifold. If $\mathrm{Ham}_c(M,L)$ is the group of compactly supported Hamiltonian diffeomorphisms of M that leave L invariant, then

$$\mathrm{R} : \widetilde{\mathrm{Ham}_c}(\mathrm{M},\mathrm{L}) \to \mathbb{R}$$

$$\{\phi_t\} \longmapsto \int_0^1 \int_M H_t \omega^n dt \ ,$$

where H_t is given by $i_{X_t}\omega = dH_t$ and $\frac{d}{dt}\phi_t = X_t \circ \phi_t$, is the relative Calabi homomorphism. That this homomorphism is a well-defined surjective homomorphism can be proved almost the same as the absolute case (see for example [**1**] p.103).

Similarly, the relative Calabi homomorphism can be defined for compact manifolds. Namely, if $\widetilde{\mathrm{Ham}}_{U,U\cap L}(M,\omega)$ denotes the universal cover of Hamiltonian diffeomorphisms supported in U that leave the Lagrangian submanifold L invariant then

$$\mathrm{R}_{\mathrm{U},\mathrm{U}\cap\mathrm{L}} : \widetilde{\mathrm{Ham}}_{\mathrm{U},\mathrm{U}\cap\mathrm{L}}(\mathrm{M},\omega) \to \mathbb{R}$$

$$\{\phi_t\} \longmapsto \int_0^1 \int_M H_t (\omega)^n dt$$

is again a surjective homomorphism.

Remark 2.2. Let $\mathrm{R} : \tilde{\mathrm{G}} \to \mathbb{R}$ denote any of the above versions of the Calabi homomorphisms in the universal cover setting. We use the same notation for the induced homomorphisms for the underlying groups. Namely, if Λ denotes the image of $\pi_1(G)$ under R, then

$$\mathrm{R} : \mathrm{G} \to \mathbb{R}/\Lambda$$

is a well-defined homomorphism.

2.2. Relative Weinstein Charts. Let $\psi \in \mathrm{Symp}_0(M,L)$ be sufficiently C^1-close to the identity. Similar to the absolute case, there corresponds a closed 1-form $\sigma = C(\psi) \in \Omega^1(M)$ defined by $\Psi(graph(\psi)) = graph(\sigma)$. Here $\Psi : \mathcal{N}(\Delta) \to \mathcal{N}(M_0)$ is a fixed symplectomorphism between the tubular neighborhoods of the Lagrangian submanifolds diagonal ($\Delta \subset (M \times M, (-\omega) \oplus \omega)$) and the zero section ($M_0 \subset (T^*M, \omega_{can})$) of the cotangent bundle with $\Psi^*(\omega_{can}) = (-\omega) \oplus \omega$. Note that since $\psi \in \mathrm{Symp}_0(M,L)$ the corresponding 1-form vanish on TL, i.e. $\sigma|_{TqL} = 0$ for any $q \in L$. Here ω_{can} denotes the canonical symplectic form on the cotangent bundle of a smooth manifold. See [**4**] for the absolute versions and the details. As a consequence we have the following due to Ozan:

Lemma 2.3. ([**5**]) If $\psi \in \mathrm{Symp}_0(M, L, \omega)$ is sufficiently C^1-close to the identity and $\sigma = C(\psi_t) \in \Omega'(M)$ then $\psi \in \mathrm{Ham}(M, L)$ iff $[\sigma] \in \Gamma(M, L)$.

$\Gamma(M, L)$ is the relative flux group defined as the image of the fundamental group of $\mathrm{Symp}_0(M, L, \omega)$ under the flux homomorphism.

$$\Gamma(M, L) = \widetilde{Flux}(\pi_1(\mathrm{Symp}_0(M, L, \omega))) \subseteq H^1(M, L, \mathbb{R}).$$

Definition 2.4. The correspondence

$$C : \mathrm{Symp}_0(M, L, \omega) \to Z^1(M, L)$$
$$h \longmapsto C(h)$$

is called a Weinstein chart of a neighborhood of $id_M \in \mathrm{Symp}_0(M, L, \omega)$ into a neighborhood of zero in the set of closed 1-forms that vanish on TL. The form $C(h)$ is called a (relative) Weinstein form.

With these definitions in mind we have the following. Compare the absolute version in [**1**].

Lemma 2.5. Let (M, ω) be a symplectic manifold, L a Lagrangian submanifold. For any $h \in \mathrm{Ham}(M, L)$ there exists finitely many hamiltonian diffeomorphisms $h_i \in \mathrm{Ham}(M, L),\ i = 1, .., N$, such that each h_i is close to id_M to be in the domain of the Weinstein chart. Moreover $C(h_i)$ is exact for all $i = 1, .., N$.

Proof. As the above lemma suggests, every smooth path $\psi_t \in \mathrm{Ham}(M, L)$ is generated by Hamiltonian vector fields. Let h_t be any isotopy in $\mathrm{Ham}(M, L)$ to the identity such that $\frac{d}{dt} h_t = X_t(h_t)$ where $i_{X_t}\omega = df_t,\ h_0 = id_M,\ h_1 = h$ and $f_t : M \to \mathbb{R}$ are Hamiltonians for all $t \in [0, 1]$. Let N be an integer large enough so that

$$\Phi_t^i = \left[h_{\left(\frac{N-i}{N}\right)t} \right]^{-1} h_{\left(\frac{N-i+1}{N}\right)t}$$

is in the domain of the Weinstein chart. If we let $h_i = \Phi_1^i$ then we have $h = h_N h_{N-1} ... h_1$. As noted by Ozan in [**5**] the group $\Gamma(M, L)$ is countable. Therefore any continuous mapping of $[0, 1]$ into $\Gamma(M, L)$ must be constant. Hence $t \longmapsto [C(\Phi_t^i)]$ is constant and thus $[C(\Phi_t^i)] = 0$. □

2.3. The Fragmentation Lemma. By $B^1(M, L)$ denote the set of exact 1-forms that evaluates zero on TL for a Lagrangian submanifold $L \subset M$. To any smooth function $f : M \to \mathbb{R}$ that is locally constant on L, there is a continuous linear map $\sigma_{rel} : B^1(M, L) \to C_L^\infty(M)$ satisfying $\omega = d(\sigma_{rel}(\omega))$ for all $\omega \in B^1(M, L)$. Then there is a bounded linear functional $\tilde{f} : B^1(M, L) \to B^1(M, L)$, due to Palamadov [**6**], given by:

$$\tilde{f}(\xi) = d(f\sigma_{rel}(\xi)) \tag{2}$$

We make use of this construction in the proof of the Fragmentation Theorem:
Proof. (of Theorem 1.1) We use the notation of Lemma 2.5. By Lemma 2.5 any $h \in \mathrm{Ham}(M, L)$ can be written as $h = h_1 ... h_N$ where each $h_i \in \mathrm{Ham}(M, L)$ is close to id_M to be in the domain V of the Weinstein chart

$$C : V \subset \mathrm{Symp}_0(M, L) \to C(V) \subset Z^1(M, L)$$

and such that $C(h_i)$ is exact for all $i = 1, .., N$.

Start with an open cover $\mathcal{U} = (U_i)_{i\in\mathbb{N}}$ of M and a partition of unity $\{\lambda_i\}$ subordinate to it. Let K be a compact subset of M containing the support of h. Let $\mathcal{U}_k = \{U_0, ..., U_N\}$ be a finite subcover for K such that $U_i \cap U_{i+1} \neq \emptyset$. Then consider the functions

$$\mu_0 = 0 \quad , \quad \mu_j = \sum_{i\leq j} \lambda_i$$

for $j = 1, 2, ..., N$. Note that for any $x \in K$, $\mu_N(x) = 1$ and $\mu_i(x) = \mu_{i-1}(x)$ for $x \notin U_i$.

Let $\tilde{\mu_i}$ be defined as in the Equation (2). Since this operator is bounded, there is an open neighborhood $V_0 \subset V$ of $id \in \mathrm{Symp}_c(M, L)$ with

$$\tilde{\mu_i}(C(h)) \in C(V) \text{ for all } i = 1, ..., N \text{ and } h \in V_0$$

Assume that $h \in V_0$ and define

$$\psi_i = C^{-1}(\tilde{\mu}_i(C(h))) \in \mathrm{Ham}(M, L).$$

Note that $\psi_{i-1}(x) = \psi_i(x)$ for $x \notin U_i$ since $\mu_{i-1}(x) = \mu_i(x)$ in that case. Therefore $(\psi_{i-1}^{-1}\psi_i)(x) = x$ if $x \notin U_i$. Hence, $h_i = (\psi_{i-1})^{-1}(\psi_i)$ is supported in U_i. On K we have $\mu_N = 1,\ \mu_0 = 0,\ \psi_N = h$, and $\psi_0 = id$. Therefore

$$h = \psi_N = (\psi_0^{-1}\psi_1)(\psi_1^{-1}\psi_2)...(\psi_{N-1}^{-1}\psi_N) = h_1h_2...h_N.$$

For the second statement define the isotopies $h_t^i = \psi_{i-1}(t)\psi_i(t)$, where $\psi_i(t) = C^{-1}(t\mu_i(\tilde{C}(h)))$. A classical result due to Calabi states that the Lie algebra of locally supported Hamiltonian diffeomorphisms is perfect [**2**]. Since for each t, $\dot{h_t^i}$ is a Hamiltonian vector field parallel to L, we can write $\dot{h_t^i}$ as a sum of commutators. In other words we have

$$\dot{h_t^i} = \sum_j [X_t^{ji}, Y_t^{ji}],$$

where X_t^{ji} and $Y_t^{ji}$are again Hamiltonian vector fields (not necessarily parallel to L). By the cut-off lemma below X_t^{ji} and Y_t^{ji} can be chosen to vanish outside of an open set whose closure contain U_i. If u_t^i is the unique function supported in U_i with $i_{\dot{h_t^i}}\omega = du_t^i$, then $du_t^i = \sum_j \omega(X_t^{ji}, Y_t^{ji})$ since both functions above have the same differential and both have compact supports. Therefore

$$\int_{U_i} u_t^i\omega^n = \int_M u_t^i\omega^n = \sum_j \int \omega(X_t^{ji}, Y_t^{ji})\omega^n = 0$$

implying that

$$\int_0^1 \int_{U_i} u_t^i\omega^n = R_{U_i, U_K\cap L}(h_i) = 0.$$

□

The cut-off lemma we used in the proof of the fragmentation lemma is as follows.

LEMMA 2.6. Let $\varphi_t \in \mathrm{Ham}(M, L)$ be an isotopy of a smooth symplectic manifold (M, ω) leaving a Lagrangian submanifold L invariant. Let $F \subset M$ be a closed subset and $U, V \subset M$ open subsets such that $U \subset \overline{U} \subset V$ with $\cup_{t\in[0,1]}\varphi_t(F) \subset U$. Then there is an isotopy $\overline{\varphi_t} \in \mathrm{Symp}(M, L)$ supported in V and coincides with φ_t on U.

Proof. We choose a smooth function $\lambda_t(x) = \lambda(x,t)$ which equals to 1 on $U \times [0,1]$, 0 outside of $V \times [0,1]$. Let f_t denote the family of Hamiltonians corresponding to φ_t, i.e. $i_{\dot{\varphi}_t}\omega = df_t$. Define $\overline{X}(x,t) = X_{(\lambda_t \cdot f_t)} + \partial/\partial t$ on $M \times [0,1]$, where $X_{(\lambda_t \cdot f_t)}$ is the Hamiltonian vector field given by $i_{X_{(\lambda_t \cdot f_t)}}\omega = d(\lambda_t \cdot f_t)$.The desired isotopy is obtained by integrating the vector field $\overline{X}(x,t)$. □

3. Final Remarks

(1) Ham(M, L) is not simple because of the following. Consider the sequence of groups and homomorphisms:

$$0 \longrightarrow \mathrm{Ker}\varphi \longrightarrow \mathrm{Ham(M,L)} \xrightarrow{\varphi} \mathrm{Diff}^{\infty}(\mathrm{L}) \longrightarrow 0,$$

where φ is just restriction to L. Therefore Kerφ consists of Hamiltonian diffeomorphisms of M that are identity when restricted to L. Clearly, Kerφ is a closed subgroup.

(2) The remaining parts of the Thurston and Banyaga's proofs fail to work in the relative case. In the absolute case the proof is completed by finding one smooth manifold for which the perfectness is easily shown. In both smooth and symplectic categories this is done through the torus due to a theorem by Herman [**3**]. Unfortunately the underlying KAM theory fails to apply in the relative case and thus whether Ham(M,L) is perfect remains open.

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