COLORED SIGNATURES OF 2-BRIDGE LINKS

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ABSTRACT. Given a 2-bridge link L, we study an algorithm that calculates the colored signature and multivariable Alexander polynomial of L. We find all 2-Bridge links up to 11 crossings and locate them in Thistlethwaite's link table. The splitting numbers of some links are calculated as a consequence of this identification.

1. INTRODUCTION

Let L be a link in S^3 and A be a Seifert matrix for L. The Alexander polynomial $\Delta_L(t)$ of L equals $det(A - tA^T)$ and if $\omega \in S^1 \setminus \{1\} \subset \mathbb{C}$ then $H(\omega) = (1 - \bar{\omega})(A - \omega A^T)$ defines a Hermitian matrix. The signature of H (number of positive eigenvalues minus the number of negative eigenvalues) is the signature function $\sigma_L(\omega)$, and the number of zero eigenvalues of H for ω is the nullity $\eta_L(\omega)$ of L. The case $\omega = -1$ is due to Trotter [15] and Murasugi [13] for links. The generalization to all $\omega \in S^1 \setminus \{1\}$ is done by Levine [11] and Tristram [14]. Such knot/link invariants that are defined through Seifert matrices can naturally be generalised to colored links via C-complexes which are generalized Seifert surfaces. Cimasoni and Florens [4] uses C-complexes to generate Seifert matrices which they use to define colored signatures and calculate multivariable Alexander polynomials.

The aim of this study is twofold: to compute the colored signature and Alexander polynomial of a 2-bridge link L, and find all 2-bridge links inside Thistlethwaite's link table. There are various properties of links that are computed and listed in databases, such as KnotAtlas [10], Knotilus [9], or Knotinfo [3] Identifying 2-bridge links inside the list of links will enable the information about 2-bridge links to be tied to their representatives in these databases. Thistlethwaite's [8] table of links is the most common enumeration used in the databases. The outline of the paper is as follows: In section2 we review colored links, C-complex and how signature and Alexander polynomials are calculated, following mainly [4]. Section 3 is devoted to the summary of 2-bridge links with the exposition of Murasugi's book [12]. Next we give the algorithm that calculates the colored signature of any 2-bridge link. In section 5, we give an algorithm for identifying all 2-bridge links of a certain crossing number. As an application, we calculate the splitting numbers of some 2-bridge links that have certain type of Conway normal forms in section 6.

In this study MATLAB® is used to implement the algorithms.

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2. Colored Links and their signatures

Let $L = L_1 \cup L_2 \cup \cdots \cup L_\mu$ be a μ-colored link. This means L is an oriented link in S^3 (or more generally in a homology S^3) and there is a surjective map, called the coloring, that assigns a color $\{1, \ldots, \mu\}$ to each component of L. Note that the number of components of L may be greater than μ, i.e. some components may be given the same color.

The concept of Seifert surface of a knot (or a link) is generalized as the C-complex structure for colored links.

Definition 2.1. The union of surfaces $S = S_1 \cup S_2 \cup \cdots \cup S_\mu$ is a C-complex for the colored link $L = L_1 \cup L_2 \cup \cdots \cup L_\mu$ such that,

(1) S_i is a Seifert surface for L_i for all i,
(2) if $i \neq j$, then $S_i \cap S_j$ is either empty or a union of "clasps" (Figure 1),
(3) for all $i,\ j,\ k$ pairwise distinct $S_i \cap S_j \cap S_k$ is empty.

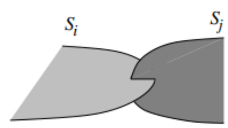

Figure 1. A clasp intersection, (figure taken from [4])

The generalization of Levine-Tristram signature to colored links is via the Seifert form on C-complexes. Let S be a C-complex for the μ-colored link L. Fix a basis for $H_1(S)$ and consider the bilinear form

$$\begin{aligned} \alpha^\epsilon : H_1(S) \times H_1(S) &\longrightarrow \quad \mathbb{Z} \\ (\mathbf{x}, \mathbf{y}) &\longmapsto \quad lk(i^\epsilon(x), y) \end{aligned}$$

where lk denotes the linking number, $\epsilon = (\epsilon_1, \ldots, \epsilon_\mu)$ is a sequence of ± 1's, $i^\epsilon(c)$ is the class that is obtained by pushing c in the ϵ_i normal direction off S_i for $i = 1, \ldots, \mu$. If we denote the matrix of α^ϵ by A^ϵ, then observe that $A^{-\epsilon} = (A^\epsilon)^T$. This means for 2^μ possible ϵ, there are $2^{\mu-1}$ matrices (up to transposition) to be calculated for the signature of L. The signature function defined on the μ-dimensional torus $T_*^\mu = (S^1 \setminus \{1\})^\mu$ is the signature of the Hermitian matrix

$$H(\omega) = \sum_\epsilon \prod_{i=1}^{\mu} (1 - \bar{\omega_i}^{\epsilon_i})\ A^\epsilon.$$

In particular for $\mu = 2$, $H(\omega) = H(\omega_1, \omega_2)$ becomes

$$H(\omega_1, \omega_2) = (1 - \bar{\omega_1})(1 - \bar{\omega_2})[A^{++} - \omega_1 A^{-+} - \omega_2 A^{+-} + \omega_1\omega_2 A^{--}]. \tag{2.1}$$

Since H is Hermitian its eigenvalues are real. The signature and nullity of H are the signature σ_L and nullity η_L of L respectively at ω. If we express the formula 2.1 as $H(\omega_1, \omega_2) = (1 - \bar{\omega_1})(1 - \bar{\omega_2})A(\omega_1, \omega_2)$, then the multivariable Alexander polynomial of L is the determinant of the matrix $A(u, v)$ up to a factor of $(u-1)(v-1)$. (See [6].) An interesting application of signature and nullity can be found in [5], where the authors use them to find an upper bound for the splitting number of links.

3. Two-Bridge Links

Two-bridge links are links with two components, which can be put into the form as in Figure 2 below. The integers $a_1, \dots, a_n$ denote the number of overcrossings (positive a_i) or undercrossings (negative a_i) with respect to an orientation. The

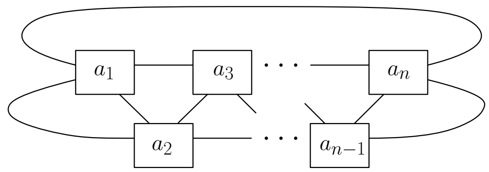

Figure 2. Figure: A 2-bridge knot or link.

2-bridge links are also called rational links, since they can be classified by rational numbers. If $gcd(p, q) = 1$, then the rational number given by the continued fraction

$$\frac{p}{q} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \frac{1}{a_n}}}$$

corresponds to a 2-bridge knot or link. This link can be denoted by this rational number as $L(p, q)$ or with the Conway normal form $C(a) = C(a_1, \dots, a_n) = [a_1, \dots, a_n]$. Note that, for $\frac{p}{q}$ to represent a link, p must be even. Continued fraction expression of a rational number is not unique and in this study we choose a normal form in which each a_i is even for a more systematic calculation of the signature. In this case its Conway normal form with even coordinates $C(a) = C(2b_1, \dots, 2b_m)$ will have odd number of coordinates. See [12] for more details on 2-bridge links.

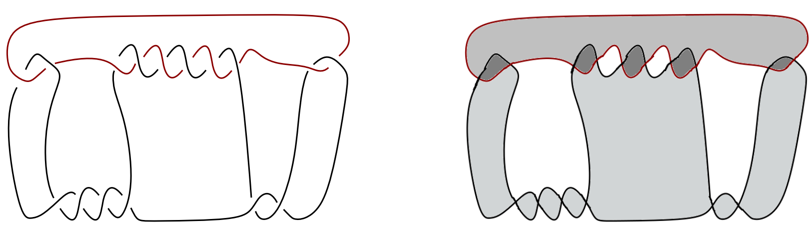

Figure 3. The link $C(2, 4, 6, 2, 2)$ and its natural C-complex

4. Calculating the Signature Function

Let $L = \frac{p}{q}$ be a 2-bridge link with the Conway normal form $L = C(a)$. We rewrite this with even coordinates only as $L = C(2a_1, 2b_1, 2a_2, \dots, 2b_{n-1}, 2a_n)$. This allows us to unify the clasp intersections into a single type and therefore the natural C-complex of L takes a form that the linking numbers for a given basis are systematically calculated.

As Figure 4 suggests, for a basis of the first homology of a link with normal form $L = C(2a_1, 2b_1, 2a_2, \dots, 2b_{n-1}, 2a_n)$ there corresponds $|a_i| - 1$ generators of type α for each coordinate a_i and $n-1$ generators of type β. In total $H_1(S)$ is generated by $k = (|a_1| + \cdots + |a_n|) - n + (n-1) = (|a_1| + \cdots + |a_n|) - 1$ generators. Indexing these

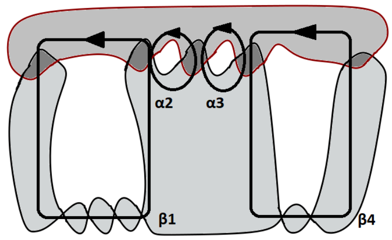

FIGURE 4. A basis for the first homology of the C-complex of the link $C(2,4,6,2,2)$

generators from left to right, we see that the only possible nonzero linking numbers are those belonging to consecutive generators and self-linkings. Corresponding k-by-k Seifert matrix A^{++} is diagonal having the only nonzero entries $lk(\beta_i^{++}, \beta_i) = -b_i$.

For A^{+-} we calculate $lk(\beta_i^{+-}, \beta_i)$ as below, depending on the signs of the neighboring coordinates a_j and a_{j+1}:

$$B_i = lk(\beta_i^{+-}, \beta_i) = \begin{cases} -1 - b_i & \text{for } (+, 2b_i, +) \\ -b_i & \text{for } (+, 2b_i, -) \\ -b_i & \text{for } (-, 2b_i, +) \\ 1 - b_i, & \text{for } (-, 2b_i, -) \end{cases}$$

and $lk(\alpha_i^{+-}, \alpha_i) = \pm 1$ which is the opposite of the sign of the coordinate a_j that this generator is resulting from. Above the diagonal we have $lk(\gamma_i^{+-}, \gamma_{i+1}) = \pm 1$ which is equal to the sign of the coordinate a_j that creates the common clasp intersection of these consecutive generators, regardless of their type being α or β.

$$A^{++} = \begin{bmatrix} 0 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & 0 & & & & & & & \\ & & & -b_1 & & & & & & \\ & & & & 0 & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & 0 & & & \\ & & & & & & & -b_2 & & \\ & & & & & & & & 0 & \\ & & & & & & & & & \ddots \\ & & & & & & & & & & 0 \\ & & & & & & & & & & & \ddots \\ & & & & & & & & & & & & 0 \end{bmatrix} \quad \begin{array}{l} \}\, |a_1| - 1 \; times \\ \}\, |a_2| - 1 \; times \\ \}\, |a_n| - 1 times \end{array}$$

$$A^{+-} = \begin{bmatrix} -1 & \pm 1 & & & & & & & & \\ & \ddots & \ddots & & & & & & & \\ & & -1 & \pm 1 & & & & & & \\ & & & B_1 & \pm 1 & & & & & \\ & & & & -1 & \pm 1 & & & & \\ & & & & & \ddots & \ddots & & & \\ & & & & & & -1 & \pm 1 & & \\ & & & & & & & B_2 & \pm 1 & \\ & & & & & & & & \ddots & \ddots \\ & & & & & & & & & \ddots & \ddots \\ & & & & & & & & & & -1 & \pm 1 \\ & & & & & & & & & & & \ddots & \pm 1 \\ & & & & & & & & & & & & -1 \end{bmatrix} \quad \begin{array}{l} \}\, |a_1| - 1 \; times \\ \}\, |a_2| - 1 \; times \\ |a_n| - 1 \; times \, \{ \end{array}$$

FIGURE 5. The matrix A^{++} is diagonal with $n-1$ nonzero entries corresponding to $lk(\beta_i^{++}, \beta_i) = -b_i$ and A^{+-} is bidiagonal.

Applying the calculations in Figure 5 to the link $C(2,4,6,2,2)$ with the basis depicted in Figure 4, we get the Seifert matrices as in Figure 6 for this link.

The matrices are, then, plugged in Equation 2.1 to get the Hermitian matrix $H(\omega_1, \omega_2)$ whose signature gives the colored signature $\sigma_L(\omega_1, \omega_2)$ of the link L. The Alexander polynomial is calculated as the determinant of $A(u, v) = [A^{++} - uA^{-+} - vA^{+-} + uvA^{--}]$. For the link $C(2,4,6,2,2)$ we calculate the Alexander polynomial as $2u^4v^2 - 7u^4v + 6u^4 + 2u^3v^3 - 11u^3v^2 + 17u^3v - 7u^3 + 2u^2v^4 - 11u^2v^3 +$

$$A^{++} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad A^{+-} = \begin{bmatrix} -3 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

FIGURE 6. Seifert matrices of the link $C(2, 4, 6, 2, 2)$

$19u^2v^2 - 11u^2v + 2u^2 - 7uv^4 + 17uv^3 - 11uv^2 + 2uv + 6v^4 - 7v^3 + 2v^2$. The signature and nullity functions are constant on the complement of the zeros of the Alexander polynomial. If the domain of the open torus is sketched as a square, then the values of the signature function for the link $C(2, 4, 6, 2, 2)$ is given in Figure 7.

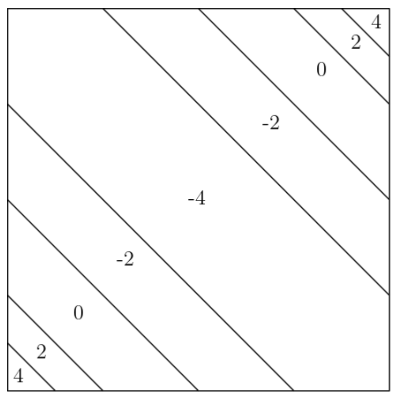

FIGURE 7. Colored signature values of the link $C(2, 4, 6, 2, 2)$ on the open torus.

In particular $\sigma_L(-1, -1)$ equals the classical signature σ_L of the link. As a consequence of above calculations we get the following result on the signature of a particular family of 2-bridge links.

Theorem 4.1. *Let L be a 2-bridge link with Conway form* $C(2a_1, 2b_1, 2a_2, \dots, 2b_{n-1}, 2a_n)$ *where* $b_1, \cdots, b_{n-1} > 0$. *Then, the signature* $\sigma_L(-1, -1) = \sigma_L = 1 - (|a_1| + \cdots + |a_n|)$ *and nullity* $\eta_L(-1, -1) = \eta_L = 0$.

Proof. The statement of the theorem is equivalent to negative definiteness of the Hermitian matrix $H(-1, -1)$ for this class of links. H is a tridiagonal symmetric matrix as in the form o Figure 8. The diagonal entry β_i is equal to $2(B_i - b_i)$ where B_i is as calculated above. In any case if $b_i > 0$ we have $\beta_i \leq -2$. As a consequence of Sylvester's Criterium, H is negative definite, if all its principal minors are negative definite. Then the result follows, for instance, from Proposition 2.1 of [1]. □

5. IDENTIFYING AND LISTING THE LINKS

In this section, we outline an algorithm to find all 2-bridge links with n crossings. The results for links up to 11 crossing are listed in section 7. These links are matched with their Thistlethwaite's Id. This will allow other data about these links to be relatable to their Conway normal forms. For a similar study on the

$$H(-1,-1) = 4\begin{bmatrix} -2 & \pm1 & & & & & & & \\ \pm1 & -2 & \pm1 & & & & & & \\ & \ddots & \ddots & \ddots & & & & & \\ & & \pm1 & -2 & \pm1 & & & & \\ & & & \pm1 & \beta_1 & \pm1 & & & \\ & & & & \pm1 & -2 & \pm1 & & \\ & & & & & \ddots & \ddots & \ddots & \\ & & & & & & \ddots & \ddots & \ddots \\ & & & & & & & \pm1 & -2 \end{bmatrix}$$

FIGURE 8. The Levine-Tristram signature of L is the signature of H

2-bridge knots, see De Wit's paper[7]. The following theorems, due to Schubert [16], are used frequently in this study to eliminate equivalent links from the list of possible combinations.

Theorem 5.1. ***(Schubert) (Theorem 9.3.3 of*** [12]***)*** *The 2-bridge links $L(p,q)$ and $L(p',q')$ are equivalent as unoriented links if and only if $p = p'$ and $qq' \equiv 1\ (mod\ p)$.*

Theorem 5.2. ***(Schubert) (Theorem 9.4.1 of*** [12]***)*** *If the orientation of one component of the 2-bridge link $L(p,q)$, where both $p > 0$ and $q > 0$, is reversed the resulting link is equivalent to $L(p, q-p) = L^*(p, p-q)$.*

In Example 5.3 below, we exhibit how these theorems are used to find equivalent links of 7 crossings (up to orientation). In order to find and identify 2-bridge links up to a certain number of crossings n:

(1) Find all permutations of positive integers that add up to n,
(2) Calculate the rational number $\frac{p}{q}$ for each such permutation,
(3) Rule out the permutations giving knots instead of links, by checking the parity of p,
(4) Pick one representative of permutations with equal continued fraction or equivalent link according to Schubert's criteria (Theorem 5.1 and 5.2),
(5) Identify the Gauss code of the link by looking at its Conway form,
(6) Locate it in Thistlethwaite's link table.

The Thistlethwaite's link table lists links up to orientations and mirror images. This means in the tables below, a link may represent 2 or 4 nonequivalent links. If the link has a palindromic Conway form, then there is only one possible orientation. In this case its mirror image is the second link represented by the same DT Id. Otherwise, there are two orientations for both of the link and its mirror separately.

Example 5.3. In Table 1, we list all 20 possible 2-bridge links with 7 crossings and explain why these are represented by only 3 links in the corresponding table of links with 7 crossings. Consider the links E_1 to E_8. The links with continued fractions leading to equal rational number are equivalent, hence we have $E_1 = E_2$, $E_3 = E_4$, $E_5 = E_6$ and $E_7 = E_8$. By Theorem 5.1, $E_1 = E_3$ since $3{\cdot}5 \equiv 1\ (mod\ 14)$. Similarly

ID	Link (p,q)	Conway Form		ID	Link (p,q)	Conway Form
E_1	$(14,3)$	$[4,1,1,1]$		E_{11}	$(16,9)$	$[1,1,3,1,1]$
E_2	$(14,3)$	$[4,1,2]$		E_{12}	$(16,9)$	$[1,1,3,2]$
E_3	$(14,5)$	$[2,1,3,1]$		E_{13}	$(18,5)$	$[3,1,1,1,1]$
E_4	$(14,5)$	$[2,1,4]$		E_{14}	$(18,5)$	$[3,1,1,2]$
E_5	$(14,9)$	$[1,1,1,3,1]$		E_{15}	$(18,7)$	$[2,1,1,2,1]$
E_6	$(14,9)$	$[1,1,1,4]$		E_{16}	$(18,7)$	$[2,1,1,3]$
E_7	$(14,11)$	$[1,3,1,1,1]$		E_{17}	$(18,11)$	$[1,1,1,1,2,1]$
E_8	$(14,11)$	$[1,3,1,2]$		E_{18}	$(18,11)$	$[1,1,1,1,3]$
E_9	$(16,7)$	$[2,3,1,1]$		E_{19}	$(18,13)$	$[1,2,1,1,1,1]$
E_{10}	$(16,7)$	$[2,3,2]$		E_{20}	$(18,13)$	$[1,2,1,1,2]$

TABLE 1. All possible combinations of 7 crossings

$E_5 = E_7$. These equivalences assume that the links are not oriented. Suppose E_3 is given the standard orientation. According to Theorem 5.2, if the orientation of one of the components of E_3 is reversed the resulting link is equivalent (as oriented links) to E_5. Since Thistlethwaite's link table lists links up to orientations all these 8 links are represented by one link, namely L_7A_6. Similarly the links E_9 to E_{12} and E_{13} to E_{20} are represented by L_7A_4 and L_7A_5, respectively.

6. Splitting Numbers

Although the classification of 2-bridge links is complete, various characteristics or local properties of these links are actively studied. In this section we will mention one of such invariants, namely the splitting number of a link.

The splitting number, $sp(L)$, of the link L is defined to be the minimum number of crossing changes between different components of L to convert L into a split link. For more information about splitting numbers see [5] and [2] and references therein. In [2], authors calculate the splitting numbers of links up to 9 crossings. They use 5 methods based on covering properties or Alexander invariants, case by case, for determining the splitting numbers. The other study [5], due to Cimasoni et. al., calculates the splitting number by looking at the signature and nullity of the links. As a consequence of their main result, the following theorem is about the splitting number of certain 2-bridge links:

Theorem 6.1. ***(Theorem 4.7 of*** [5]***)*** *The splitting number of the 2-bridge link* $C(2a_1, b_1, \ldots, 2a_{n-1}, b_{n-1}, 2a_n)$ *is* $a_1 + a_2 + \cdots + a_n$*, where all* a_i *and* b_i *are positive integers.*

Our calculations below reveal which alternating two component links are 2-bridge links. Therefore one can also calculate the splitting numbers of the following links, which turn out to be 2-bridge links with the desired Conway form of Theorem 6.1:

L_5A_1, L_6A_1, L_7A_4, L_7A_6, L_8A_6, L_8A_8, L_8A_{11}, L_9A_{18}, L_9A_{26}, L_9A_{30}, L_9A_{36}, L_9A_{40}

Besides the above links that already appear with splitting numbers in [2], we calculate the splitting numbers of those links in Table 2 with 10 and 11 crossings that fit into the Conway form of Theorem 6.1:

Link	$sp(L)$		Link	$sp(L)$		Link	$sp(L)$
$L_{10}A_{48}$	2		$L_{11}A_{132}$	2		$L_{11}A_{299}$	4
$L_{10}A_{64}$	3		$L_{11}A_{194}$	3		$L_{11}A_{312}$	4
$L_{10}A_{75}$	3		$L_{11}A_{206}$	3		$L_{11}A_{319}$	4
$L_{10}A_{87}$	3		$L_{11}A_{222}$	3		$L_{11}A_{360}$	5
$L_{10}A_{89}$	4		$L_{11}A_{263}$	4		$L_{11}A_{372}$	5
$L_{10}A_{98}$	4		$L_{11}A_{278}$	4			
$L_{10}A_{102}$	4		$L_{11}A_{289}$	4			

TABLE 2. Splitting numbers of some 2-bridge links

We note that the splitting number of $L_{11}A_{372}$ was also calculated in [5] (Example 4.4). In their work, the authors are not making use of the fact that Theorem 6.1 is also applicable to this link.

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7. TWO-BRIDGE LINKS UP TO 11 CROSSINGS

The Link $\frac{p}{q}$ or (p,q)	Conway Form	Thistlethwaite's Id
(4,1)	[4] or [3,1]	L_4A_1
(8,3)	[2,1,2]	L_5A_1
(6,1)	[6] or [5,1]	L_6A_3
(10,3)	[3,3]	L_6A_2
(12,5)	[2,2,2]	L_6A_1
(14,5)	[2,1,4]	L_7A_6
(16,7)	[2,3,2]	L_7A_4
(18,5)	[3,1,1,2]	L_7A_5

TABLE 3. 2-Bridge Links of 4,5,6 and 7 Crossings

The Link $\frac{p}{q}$ or (p,q)	Conway Form	Thistlethwaite's Id
(8,1)	[8] or [7,1]	L_8A_{14}
(16,3)	[5,2,1]	L_8A_{12}
(20,9)	[2,4,2]	L_8A_6
(22,5)	[4,2,2]	L_8A_{11}
(24,7)	[3,2,3]	L_8A_{13}
(26,7)	[3,1,2,2]	L_8A_{10}
(30,11)	[2,1,2,1,2]	L_8A_8
(34,13)	[2,1,1,1,1,2]	L_8A_9

TABLE 4. 2-Bridge Links of 8 Crossings

The Link $\frac{p}{q}$ or (p,q)	Conway Form	Thistlethwaite's Id
(20,7)	[2,1,6]	L_9A_{36}
(24,5)	[4,1,4]	L_9A_{40}
(24,11)	[2,5,2]	L_9A_{18}
(28,11)	[2,1,1,5]	L_9A_{39}
(30,7)	[4,3,2]	L_9A_{30}
(32,7)	[4,1,1,3]	L_9A_{38}
(34,9)	[3,1,3,2]	L_9A_{25}
(36,11)	[3,3,1,2]	L_9A_{34}
(40,11)	[3,1,1,1,3]	L_9A_{35}
(44,13)	[3,2,1,1,2]	L_9A_{37}
(46,17)	[2,1,2,2,2]	L_9A_{26}
(50,19)	[2,1,1,1,2,2]	L_9A_{27}

TABLE 5. 2-Bridge Links of 9 Crossings

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The Link $\frac{p}{q}$ or (p,q)	Conway Form	Thistlethwaite's Id
(10,1)	[10] or [9,1]	$L_{10}A_{118}$
(22,3)	[7,3]	$L_{10}A_{114}$
(26,5)	[5,5]	$L_{10}A_{120}$
(28,13)	[2,6,2]	$L_{10}A_{48}$
(32,5)	[6,2,2]	$L_{10}A_{98}$
(38,9)	[4,4,2]	$L_{10}A_{75}$
(40,7)	[5,1,2,2]	$L_{10}A_{97}$
(40,9)	[4,2,4]	$L_{10}A_{102}$
(42,11)	[3,1,4,2]	$L_{10}A_{73}$
(42,13)	[3,4,3]	$L_{10}A_{115}$
(48,11)	[4,2,1,3]	$L_{10}A_{100}$
(48,17)	[2,1,4,1,2]	$L_{10}A_{89}$
(52,11)	[4,1,2,1,2]	$L_{10}A_{99}$
(56,15)	[3,1,2,1,3]	$L_{10}A_{101}$
(56,17)	[3,3,2,2]	$L_{10}A_{94}$
(58,17)	[3,2,2,3]	$L_{10}A_{116}$
(60,13)	[4,1,1,1,1,2]	$L_{10}A_{93}$
(62,23)	[2,1,2,3,2]	$L_{10}A_{64}$
(64,19)	[3,2,1,2,2]	$L_{10}A_{96}$
(64,23)	[2,1,3,1,1,2]	$L_{10}A_{90}$
(66,25)	[2,1,1,1,3,2]	$L_{10}A_{65}$
(68,19)	[3,1,1,2,1,2]	$L_{10}A_{92}$
(70,29)	[2,2,2,2,2]	$L_{10}A_{87}$
(74,31)	[2,2,1,1,2,2]	$L_{10}A_{83}$
(76,21)	[3,1,1,1,1,1,2]	$L_{10}A_{88}$
(80,31)	[2,1,1,2,1,1,2]	$L_{10}A_{91}$

TABLE 6. 2-Bridge Links of 10 Crossings

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The Link $\frac{p}{q}$ or (p,q)	Conway Form	Thistlethwaite's Id
$(26,3)$	$[8,1,2]$	$L_{11}A_{360}$
$(32,15)$	$[2,7,2]$	$L_{11}A_{132}$
$(34,5)$	$[6,1,4]$	$L_{11}A_{372}$
$(38,5)$	$[7,1,1,2,]$	$L_{11}A_{367}$
$(44,7)$	$[6,3,2]$	$L_{11}A_{278}$
$(46,7)$	$[6,1,1,3,]$	$L_{11}A_{364}$
$(46,11)$	$[4,5,2,]$	$L_{11}A_{206}$
$(50,9)$	$[5,1,1,4,]$	$L_{11}A_{365}$
$(50,13)$	$[3,1,5,2,]$	$L_{11}A_{192}$
$(52,9)$	$[5,1,3,2,]$	$L_{11}A_{275}$
$(54,17)$	$[3,5,1,2,]$	$L_{11}A_{355}$
$(56,13)$	$[4,3,4]$	$L_{11}A_{319}$
$(58,11)$	$[5,3,1,2]$	$L_{11}A_{371}$
$(62,11)$	$[5,1,1,1,3]$	$L_{11}A_{358}$
$(62,13)$	$[4,1,3,3]$	$L_{11}A_{369}$
$(64,15)$	$[4,3,1,3]$	$L_{11}A_{302}$
$(70,13)$	$[5,2,1,1,2]$	$L_{11}A_{362}$
$(72,19)$	$[3,1,3,1,3]$	$L_{11}A_{298}$
$(74,23)$	$[3,4,1,1,2]$	$L_{11}A_{368}$
$(76,23)$	$[3,3,3,2]$	$L_{11}A_{260}$
$(76,27)$	$[2,1,4,2,2]$	$L_{11}A_{263}$
$(78,17)$	$[4,1,1,2,3]$	$L_{11}A_{366}$
$(78,29)$	$[2,1,2,4,2]$	$L_{11}A_{194}$
$(80,17)$	$[4,1,2,2,2]$	$L_{11}A_{312}$
$(82,23)$	$[3,1,1,3,3]$	$L_{11}A_{356}$
$(82,31)$	$[2,1,1,1,4,2]$	$L_{11}A_{196}$
$(84,19)$	$[4,2,2,1,2]$	$L_{11}A_{299}$
$(84,25)$	$[3,2,1,3,2]$	$L_{11}A_{271}$
$(86,25)$	$[3,2,3,1,2]$	$L_{11}A_{361}$
$(88,19)$	$[4,1,1,1,2,2]$	$L_{11}A_{280}$
$(92,21)$	$[4,2,1,1,1,2]$	$L_{11}A_{305}$
$(92,33)$	$[2,1,3,1,2,2]$	$L_{11}A_{264}$
$(94,39)$	$[2,2,2,3,2]$	$L_{11}A_{222}$
$(98,27)$	$[3,1,1,1,2,3]$	$L_{11}A_{359}$
$(98,41)$	$[2,2,1,1,3,2]$	$L_{11}A_{221}$
$(100,27)$	$[3,1,2,2,1,2]$	$L_{11}A_{297}$
$(100,39)$	$[2,1,1,3,2,2]$	$L_{11}A_{284}$
$(104,29)$	$[3,1,1,2,2,2]$	$L_{11}A_{272}$
$(106,31)$	$[3,2,2,1,1,2]$	$L_{11}A_{363}$
$(108,29)$	$[3,1,2,1,1,1,2]$	$L_{11}A_{300}$
$(112,31)$	$[3,1,1,1,1,2,2]$	$L_{11}A_{262}$
$(112,41)$	$[2,1,2,1,2,1,2]$	$L_{11}A_{289}$
$(116,45)$	$[2,1,1,2,1,2,2]$	$L_{11}A_{266}$
$(128,47)$	$[2,1,2,1,1,1,1,2]$	$L_{11}A_{247}$
$(144,55)$	$[2,1,1,1,1,1,1,1,2]$	$L_{11}A_{248}$

TABLE 7. 2-Bridge Links of 11 Crossings