Mat 355E Topology Lecture Notes

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Contents

Preface

These notes are based on the classical book 'Topology' by Munkres which we use as the main textbook in İTÜ. I also benefited from my own notes I took as a student in METU.

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Chapter 1

Preliminaries

1.1 Set Theory and Logic

Let us recall some basic definitions and notations from set theory:

- $A \cup B = \{x | x \in\ A \ or\ x \in\ B\}$ union of A and B.
- $A \cap B = \{x | x \in\ A \ and\ x \in\ B\}$ intersection of A and B.
- $\emptyset = \{\}$ empty set.
- A and B are disjoint if $A \cap B = \emptyset$.
- $p \implies q$: if p is true then q is true.
- "not q" is called the negation of q.
- contrapositive of "$p \implies q$" is "not $q \implies$ not p"
- A statement and its contrapositive are equivalent.
- "$p \iff q$" or "p if and only if q" means "$p \implies q$ and $q \implies p$". In this case:

  * $p \implies q$ is called the "'only if"' part or "sufficiency" (p is sufficient for q)

  ** $q \implies p$ or $p \Leftarrow q$ is the "'if "' part or "necessity." (p is necessary for q)

Example 1.1.1. *"For every $x \in A$, the statement P holds."*
Its negation:
"There is at least one $x \in A$ such that the statement P does not hold."

Example 1.1.2. *"Every cover has a finite subcover."*
Its negation:
"There is at least one cover that has no finite subcover."

Definition 1.1.3. $A \setminus B = A - B = \{x | x \in \ A \text{ and } x \notin B\}$ *is called the set difference of* A *and* B.

1. $A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$
2. $A - (A - B) = A \cap B$
3. $A \subset B \iff A - B = \emptyset$
4. $A \cap (B - C) = (A \cap B) - (A \cap C)$
   $A \triangle B = (A - B) \cup (B - A)$ is symmetric difference of A and B.
   $= (A \cup B) - (A \cap B)$
5. $A - (B \cup C) = (A - B) \cap (A - C)$ (De Morgan's Law)
6. $A - (B \cap C) = (A - B) \cup (A - C)$ (De Morgan's Law)
7. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
8. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Remark 1.1.4. *To prove an equality* $A = B$ *of sets, we must prove both* $A \subseteq B$ *and* $B \subseteq A$.

Proof. (of 5.) ($\subseteq$) Let $x \in A - (B \cup C)$. Then, $x \in A$ and $x \notin (B \cup C)$.
$\implies x \in A$ and $x \notin B$ and $x \notin C$

($\supseteq$) Let $x \in (A - B) \cap (A - C)$.Then, $x \in (A - B)$and $x \in (A - C)$
$\implies x \in A$ and $x \notin B$ and $x \notin C \implies x \in A$ and $x \notin (B \cup C)$
$\implies x \in A - (B \cap C)$. □

1.2 Functions and Relations

Definition 1.2.1. *Cartesian product of the sets A and B,* $A \times B = \{(a, b) | a \in A,\ b \in B\}$, *is a set of ordered pairs.*

Sometimes we use $a \times b$ instead of (a, b) to denote ordered pairs.

Definition 1.2.2. *A function* $f : A \to B$ *is a rule which assigns a unique element of* B *to every element of* A.

Functions are subsets of Cartesian products.i.e. $\{(a, f(a)|a \in A\} \subset A \times B$ The uniqueness condition in the above definition is also expressed as f being well-defined : $a_1 = a_2 \implies f(a_1) = f(a_2)$

Definition 1.2.3. *1. If $f : A \to B$ and $A_0 \subset A$ then we define the restriction of f to A_0 as*

$$f|_{A_0} : A_0 \to B \quad \textit{with the same rule as } f\textit{: } a \mapsto f(a)$$

2. *$f : A \to B$ is one-to-one (injective) if "$f(a_1) = f(a_2)$ implies $a_1 = a_2$"*

3. *$f : A \to B$ is onto (surjective) if "for every $b \in B$ there is $a \in A$ such that $f(a) = b$"*

4. *f is bijective if it is 1-1 and onto.*

5. *Let $f : A \to B$, $A_0 \subset A$, and $B_0 \subset B$*
   $f(A_0) = \{b|b = f(a)$ for some $a \in A_0\}$: image of A_0
   $f^{-}1(B_0) = \{a|f(a) \in B_0\}$: inverse image of B_0

Remark 1.2.4. *f need not be invertible in order the inverse image of B_0 to be defined.*

Lemma 1.2.5. *1. $A_0 \subset f^{-1}(f(A_0))$*

2. *$A_0 = f^{-1}(f(A_0))$ only if f is 1-1.*

3. *$f(f^{-1}(B_0)) \subset B_0$*

4. *$f(f^{-1}(B_0)) = B_0$ only if f is onto.*

Example 1.2.6. *Let $f(x) = 3x^2 + 2$. Then f is not 1-1, and not onto as a function $f : \mathbb{R} \to \mathbb{R}$*

$f^{-1}(f([0, 1])) = [-1, 1]$ implying that f is not 1-1
$f(f^{-1}([0, 5])) = [2, 5]$ implying that f is not onto.

1.2.1 Relations

A relation $\sim$ on a set A is a subset C of $A \times A$
and $x \sim y$ means $(x, y) \in C \subset A \times A$ and we read " x is related to y".

- A relation $\sim$ on A is an equivalence relation if

    1. (Reflexive) $x \sim x$ for every $x \in A$.

2. (Symmetry) If $x \sim y$ then $y \sim x$ for every $x, y \in A$.
3. (Transitivity) If $x \sim y$ and $y \sim z$ then $x \sim z$.

- If $\sim$ is an equivalence relation on A, then
  $E = \{y | x \sim y\}$ is called the equivalence class of x.

Lemma 1.2.7. *If E_1 and E_2 are equivalence classes of $\sim$, then either $E_1 = E_2$ or $E_1 \cap E_2 = \emptyset$. (Equivalence classes are either identical or disjoint)*

Proof. Let $x \in E_1 \setminus E_2$ and $y \in E_1 \cap E_2$ (WLOG)
Then, $x \in E_1$ and $y \in E_1$ implying that $x \sim y$
Since $y \in E_2$ and $x \sim y$ we have $x \in E_2$ which is a contradiction. □

Definition 1.2.8. *A family of disjoint sets whose union gives A is said to be a partition of A.*

Example 1.2.9. $A = \{a, b, c, d\}$
$P = \{\{a, b\}, \{c\}, \{d\}\}$ *is a partition for A.*

Lemma 1.2.10. *If $\sim$ is an equivalence relation on A, then its equivalence classes partition A.*

Definition 1.2.11. *A relation $\subset$ on A is an ordering relation if*

i) For every $x, y \in A$, ¨ $x \neq y$ implies $x \subset y$ or $y \subset x$ ¨

ii) There is no $x \in A$ such that $x \subset x$

iii) If $x \subset y$ and $y \subset z$ then $x \subset z$

Example 1.2.12. $x < y$ *on $\mathbb{R}$ is an ordering relation.*

Definition 1.2.13. *Let X be a set and $<$ be an ordering relation on X.
If $a < b$, then $(a, b) = \{x | a < x < b\}$ is called an open interval.*

Definition 1.2.14. *Let $<_A$ and $<_B$ be ordering relations on A and B respectively.
On $A \times B$ define $a_1 \times b_1 < a_2 \times b_2$ if ¨ $a_1 <_A a_2$ or $a_1 = a_2$ and $b_1 <_B b_2$ ¨ as the dictionary order on $A \times B$.*

1.3 Countability

Definition 1.3.1 (*Countabilty*)**.** *A set A is said to be countable if there is a $1-1$ and onto function*

$$f : \mathbb{Z}_+ \to A$$

(or equivalently $f : A \to \mathbb{Z}_+$) . $\mathbb{Z}_+ = \mathbb{N} = \{1, 2, 3, ...\}$
A subset of a countable set is countable.

Theorem 1.3.2. *For the set $B \neq \emptyset$ the following are equivalent:*

1. *There is an onto function $f : \mathbb{Z}_+ \to B$*
2. *There is a $1-1$ function $g : B \to \mathbb{Z}_+$*
3. *B is countable.*

Theorem 1.3.3. *$\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable.*

Proof. $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{Z}_+$ defined by $(n, m) \mapsto 2^n 3^m$ is a $1-1$function:

Suppose we have two pairs with same the image (n, m) and (p, q) $2^n 3^m = 2^p 3^q$.
If $n < p$, $3^m = 2^{p-n} 3^q$ contradiction since one side is even. If $n > p$, $3^m 2^{p-q} = 3^q$ contradiction since one side is even. So $n = p$. This gives $3^m = 3^q$. If $m < q$, then $1 = 3^{q-m} \Rightarrow q = m$. So $(n, m) = (p, q)$ and hence f is $1-1$. $\Rightarrow \mathbb{Z}_+ \times \mathbb{Z}_+$ is countable. □

Theorem 1.3.4. *Countable union of countable sets is countable.*

Proof. Let $\{A_\alpha\}_{\alpha\in J}$ be a family of countable sets where J is countable. For all $\alpha \in J$ there is an onto function $f_\alpha : \mathbb{Z}_+ \to A_\alpha$ and $g : \mathbb{Z}_+ \to J$.
Define $h : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \bigcup\limits_{\alpha\in J} A_\alpha$
$h(n, m) = f_{g(n)}(m)$ and show that h is onto.
Let $x \in \bigcup\limits_{\alpha\in J} A_\alpha$ then there is $\alpha' \in J$ such that $x \in A_{\alpha'}$. There is a $\overline{m} \in \mathbb{Z}_+$ such that $f_{\alpha'}(\overline{m}) = x$ and $\overline{n} \in \mathbb{Z}_+$ such that $g(\overline{n}) = \alpha'$. Thus $h(\overline{n}, \overline{m}) = f_{g(\overline{n})}(\overline{m}) = f_{\alpha'}(\overline{m}) = x$.
So h is onto $\Rightarrow \bigcup\limits_{\alpha\in J} A_\alpha$ is countable. □

Theorem 1.3.5. *Finite product of countable sets is countable.*

Proof. Is by induction. □

Definition 1.3.6. *Let X^ω denote the infinite product of X with itself*

$$X^\omega = \{(x_1, x_2, x_3, ..., x_n, ...)|x_i \in X, i \in \mathbb{Z}\}$$

Theorem 1.3.7. *Let $X = \{0, 1\}$. X^ω is uncountable.*

Proof. There is no onto function $g : \mathbb{Z}_+ \to X^\omega$
Let $g(n) = (x_{n1}, x_{n2}, x_{n3}, ..., x_{nm}, ...)$ where x_{ij}'s are either 0 or 1. Define

$$y_n = \begin{cases} 0 & x_{nn} = 1 \\ 1 & x_{nn} = 0 \end{cases}$$

If $y = (y_1, y_2, y_3, ..., y_n, ...) \in X^\omega$ then there is no integer m such that $g(m) = y$.
g is not onto $\Rightarrow X^\omega$ is uncountable. $\square$

Chapter 2

Metric Spaces

2.1 Definition and First Examples

We study metric spaces to develop the concept of continuity.

Definition 2.1.1. *Let M be a set,*

$$\rho : M \times M \to \mathbb{R}$$

be a function. Then (M, ρ) is a metric space if

i) $\rho(x, y) \geq 0$, *and*

  i)* $\rho(x, y) = 0$ *if and only if* $x = y$,

ii) $\rho(x, y) = \rho(y, x)$,

iii) $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$ *(Triangle Inequality)*

In this case ρ is said to be a metric on M. If ρ does not satisfy i), then it is called a pseudo-metric on M.*

Example 2.1.2. $M = \mathbb{R}$, $\rho(x, y) = |x - y|$

i) $|x - y| \geq 0$ *(by definition) and* $|x - y| = 0$ *if and only if* $x = y$

ii) $|x - y| = |y - x|$

iii) $|x - z| = |x - y + y - z| \leq |x - y| + |y - z|$

So,$(\mathbb{R}, |, |)$ *is a metric space with the absolute value metric and called the standard or usual Euclidean metric space.*

Example 2.1.3. *$M = \mathbb{R}^n$, $\rho((x_1, ..., x_n), (y_1, ..., y_n)) = \sqrt{\sum\limits_{k=1}^{n} (x_k - y_k)^2}$*

For $n = 2$: $\rho((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)}$
ρ is the Standard metric on $\mathbb{R}^2$.

Example 2.1.4. *On $M = \mathbb{R}^2$ the function $\rho_1((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$ is called the Taxi metric on $\mathbb{R}^2$.*

Example 2.1.5. *Let $M = \mathbb{R}^2$ with $\rho_2((x_1, x_2), (y_1, y_2)) = max\{|x_1 - y_1|, |x_2 - y_2|\}$ which is the square metric on $\mathbb{R}^2$.*

Example 2.1.6. *The distance from the point $(x, y) \in \mathbb{R}$ to the origin with respect to above metrics is*

- $\rho((x, y), (0, 0)) = \sqrt{x^2 + y^2}$
- $\rho_1((x, y), (0, 0)) = |x| + |y|$
- $\rho_2((x, y), (0, 0)) = \max\{|x|, |y|\}$

Remark 2.1.7. *If (M, ρ) is a metric space and $A \subset M$ then $(A, \rho|_{A\times A})$ is also a metric space.*

$$\begin{aligned} \rho|_{A\times A} \colon A \times A &\to \mathbb{R} \\ (a_1, a_2) &\mapsto \rho(a_1, a_2). \end{aligned}$$

Example 2.1.8. *X: set , $\rho : X \times X \to \mathbb{R}$*

$$\rho(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

ρ is called the discrete metric on X.

Example 2.1.9. *For any set X, let $\rho(x, y) = 0$ for all $x, y \in X$. ρ is called the trivial (pseudo)-metric on X.*

2.2 ε-balls and continuity

Definition 2.2.1. *Let (M, ρ) and (N, σ) be metric spaces. $f : M \to N$ is continuous at $x_0 \in M$ if and only if, for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\rho(x, y) < \delta$ implies $\sigma(f(x), f(y)) < \varepsilon$.*

This definition applied to $\mathbb{R}$ with the usual absolute value metric yields the usual continuity definition from our calculus courses:

$f : \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$ iff $\forall \varepsilon > 0 \quad \exists \delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

Definition 2.2.2. *Let (M, ρ) be a metric space and $x \in M$. For $\varepsilon > 0$ $B_\epsilon(x) = B(x, \epsilon) = U_\rho(x, \epsilon) = \{y \in M \mid \rho(x, y) < \epsilon\}$ is called the $\underline{\epsilon - disc}$ (or ϵ-ball) centered at x. (the open ϵ-disc)*

Example 2.2.3. *Let $M = \mathbb{R}$ and $\rho = |\cdot|$ be the absolute value metric. $U_\rho(x, \epsilon) = \{y \mid \ |x - y| < \epsilon\} = (x - \epsilon, x + \epsilon)$. That is, the ϵ-ball in the usual real line is the open interval centered at x with radius ϵ.*

Example 2.2.4. *Let $M = \mathbb{R}^2$ and ρ be the standard metric. Then, $U_\rho((x_1, x_2), \epsilon) = \{(y_1, y_2) | \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} < \epsilon\}$ is the open disc centered at (x_1, x_2) with radius ϵ.*

Example 2.2.5. *For the discrete metric on $\mathbb{R}$ we have $U_\rho(x, \frac{1}{2}) = \{x\} \quad U_\rho(x, \frac{3}{2}) = \mathbb{R}$*

Definition 2.2.6. *Let E and F be subsets of the metric space (M, ρ). $\rho(E, F) = inf\{\rho(x, y) | x \in E \quad y \in F\}$ is the <u>distance between E and F.</u>*

Example 2.2.7. *Let $E = (0, 1)$ and $F = (2, 3)$ be subsets of $(\mathbb{R}, \rho = |\cdot|)$. Then $\rho(E, F) = 1$.*

Definition 2.2.8. *$U_\rho(E, \varepsilon) = \{y \in M | \rho(E, y) < \varepsilon\}$ is the ε-disc at the subspace $E \subset M$*

Example 2.2.9. *For $E = (0, 1]$ and $\varepsilon = \frac{1}{2}$ we have $U_\rho((0, 1], \frac{1}{2}) = (\frac{-1}{2}, \frac{3}{2})$.*

Definition 2.2.10. *$f : (M, \rho) \to (N, \sigma)$ is <u>continuous</u> at $x \in M$ if for all $\epsilon > 0$ there is $\varepsilon > 0$ such that $f(U_\rho(x, \delta)) \subset U_\sigma(f(x), \varepsilon)$.*

2.3 Open Sets

Definition 2.3.1. *$E \subset (M, \rho)$ is said to be <u>open</u>, if for every $x \in E$ there is $\varepsilon > 0$ such that $U_\rho(x, \varepsilon) \subset E$.*

Example 2.3.2. *$M = \mathbb{R}^2 \quad \rho$: standard metric. The sets $A = \{(x, y) | y \neq f(x)\}$, $B = \{(x, y) | y < f(x)\}$ are open for any function $f : \mathbb{R} \to \mathbb{R}$.*

Example 2.3.3. *$M = \mathbb{R} \quad \rho = |\cdot| \quad (a, b) = \{x \in \mathbb{R} | a < x < b\}$*

Let $\varepsilon = \frac{1}{2} min\{|a - x|, |x - b|\}$, then $U_\rho(x, \varepsilon) = (x - \varepsilon, x + \varepsilon) \subset (a, b)$. So (a, b) is an open set.

Definition 2.3.4. *A set $F \subset M$ is closed iff $M - F$ is open.*
$\iff$ $[y \in M - F \Rightarrow \exists \varepsilon > 0$ *such that* $U_\rho(y, \varepsilon) \subset M - F]$
$\iff$ $[\forall \varepsilon > 0 \quad U_\rho(y, \varepsilon) \cap F \neq \emptyset \Rightarrow y \in F]$.

Example 2.3.5. *Let $F = (0, 1] \subset \mathbb{R}$. Then F is not closed. For any $\varepsilon > 0 \quad B(0, \varepsilon) \cap (0, 1] \neq \emptyset$, but $0 \notin F$.*

Example 2.3.6. *Singletons (single element set) are closed in metric spaces.*

Proof. Let $x \in (M, \rho)$ and consider $\{x\}$. If $y \in M - \{x\}$, then $y \neq x$ and let $\varepsilon = \frac{1}{2}\rho(x, y) > 0$. Since $B(y, \varepsilon) \subset M - \{x\}$, $M - \{x\}$ is an open set. So $\{x\}$ is closed. □

Theorem 2.3.7. *Let (M, ρ) be a metric space.*

i) Every union of open sets is open.

ii) Finite intersection of open sets is open.

iii) ϕ and M are open

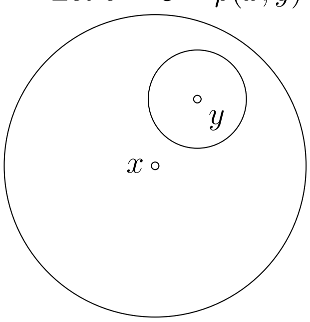
Proof. (i) Let A_α be open for any $\alpha \in \lambda$ and consider $x \in \bigcup_{\alpha\in\lambda} A_\alpha$.
$x \in \bigcup_{\alpha\in A} A_\alpha \Rightarrow$ there is $\alpha\prime$ such that $x \in A_{\alpha\prime}$ and $A_{\alpha\prime}$ is an open set.
$\Rightarrow$ There is an $\varepsilon-$disc $U_\rho(x, \varepsilon) \subset A_{\alpha\prime}$
$\Rightarrow U_\rho(x, \varepsilon) \subset \bigcup_{\alpha\in\lambda} \quad \Rightarrow \bigcup_{\alpha\in\lambda} A_\alpha$ is also an open set.

(ii) Let $A_1, ..., A_n$ be open sets. Let $x \in \bigcap_{i=1}^n A_i$.
Since A_i are open for all i, there's $\varepsilon_i > 0$ such that $U_\rho(x, \varepsilon_i) \subset A_i$
Choose $\varepsilon = min\{\varepsilon_1, \varepsilon_2, ..., \varepsilon_n\}$ then $U_\rho(x, \varepsilon) \subset A_i$ for all i.
$\Rightarrow U_\rho(x, \varepsilon) \subset \bigcap_{i=1}^n A_i \quad \Rightarrow \bigcap_{i=1}^n A_i$ is open.

□

Example 2.3.8. *For $M = \mathbb{R}$, $\rho = |\cdot|$ let $A_n = (\frac{-1}{n}, \frac{1}{n})$. Then $\bigcap_{n=1}^\infty A_n = \{0\}$ is not open. (Since for any $\varepsilon > 0 \quad U_\rho(0, \varepsilon) \not\subset \{0\}$ therefore $\{0\}$ is not open.) So infinite intersection of open sets might not be open.*

Example 2.3.9. *ε-discs are open in any metric space (M, ρ).*
Let $\delta = \varepsilon - \rho(x, y)$ or $\delta = \frac{1}{n}$'s



Show that $U_\rho(y, \delta) \subset U_\rho(x, \varepsilon)$.

Example 2.3.10. *Singletons are open in <u>discrete metric</u> spaces.*
For any point x $U_{discrete}(x, \frac{1}{2}) = \{x\} \subset \{x\}$
$\Rightarrow$ *Therefore in a discrete metric space every set is open. (Union of singletons)*
Moreover, every set is closed because it's complement is an open set.

Theorem 2.3.11. $f : (M, \rho) \to (N, \sigma)$ *is <u>continuous</u> at* $x_0 \in M$ *iff for any open set* V *with* $f(x_0) \in V \subset N$ *there is an open set* U *with* $x_0 \in U \subset M$ *st.* $f(U) \subset V$.

Proof. HOMEWORK $\square$

Chapter 3

Topological Spaces

3.1 Definition and first examples

Let X be a set and τ be a family of subsets of X such that:

i) $\phi \in \tau$ and $X \in \tau$,

ii) Any union of elements of τ is in τ,

iii) Intersection of finite number of elements of τ is in τ.

(X, τ) is called a topological space and τ is a topology on X. Elements of τ are called open sets in (X, τ). Then (X, τ) being a topological space implies:

i) ϕ, X are open,

ii) Any union of open sets is open,

iii) Intersection of finitely many open sets is open.

Example 3.1.1. *(Cofinite or finite complement topology)*
X: set, $\tau_f = \{U \subset X \mid X - U$ is finite or $U = \emptyset\}$

1) $\phi \in \tau_f$ and $X \in \tau_f$ since $X - X = \phi$ is finite.

Note: DeMorgan

$$X - (A \cap B) = (X - A) \cup (X - B)$$
$$X - (A \cup B) = (X - A) \cap (X - B)$$

2) Let $\{U_\alpha\} \in \tau_f$. Claim: $\bigcap U_\alpha \in \tau_f$
Since $X - \bigcup U_\alpha = \bigcap (X - U_\alpha)$ and intersection of finite sets is finite, claim is true. Therefore any union of sets in τ_f is also in τ_f.

3) *Let $U_1, U_2, ..., U_n \in \tau_f$. Claim: $\bigcap_{i=1}^{n} U_i \in \tau_f$*

Proof. $X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n}(X - U_i)$ and the fact that finite union of finite sets is also finite imply intersection of finitely many sets in τ_f is also in τ_f. □

Question. *Show that $\tau_c = \{U \subset X \mid X - U$ is countable or $U = \phi\}$ is a topology on any set X, called the cocountable or countable complement topology.*

Definition 3.1.2. *Let τ and $\tau^{'}$ be topologies on X. If $\tau \subset \tau^{'}$ then*
$\tau^{'}$ is stronger / finer / larger than τ
τ is weaker / coarser / smaller than $\tau^{'}$
If $\tau \subset \tau^{'}$ or $\tau^{'} \subset \tau$ then τ and $\tau^{'}$ are comparable.

Example 3.1.3. *On $X = \mathbb{Z}$, τ_f and τ_c are comparable. $\tau_f \subset \tau_c$*
$U \in \tau_f \Rightarrow X - U$ finite $\Rightarrow X - U$ is countable $\Rightarrow U \in \tau_c$

Cocountable topology is stronger than cofinite topology.

Example 3.1.4. → *On any set X, the discrete topology is the strongest topology on X.*

$\tau \subset 2^X = \tau_{discrete}$

Example 3.1.5. → *$\tau_{trivial} = \{\phi, X\}$ is the weakest topology on X.*

Example 3.1.6. *$X = \{a, b, c\}$*
$\tau_1 = \{\phi, X\}$
$\tau_2 = \{\phi, X, \{b\}\}$
$\tau_3 = \{\phi, X, \{b\}, \{a, b\}\}$
$\tau_4 = \{\phi, X, \{b\}, \{a, b\}, \{b, c\}\}$
τ_5 *= discrete topology on X*
$\tau_1 \subset \tau_2 \subset \tau_3 \subset \tau_4 \subset \tau_5$

3.2 Basis for a Topology

Definition 3.2.1. *Let X be a set and β be a family of subsets of X.*

i) For every $x \in X$ there's $B_x \in \beta$ st $x \in B_x$
(or equivalently $\bigcup_{B \in \beta} B = X$)

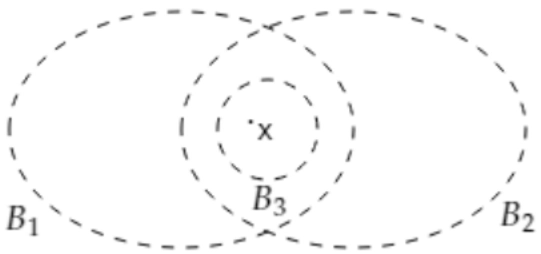
ii) For any $x \in B_1 \cap B_2$ for $B_1, B_2 \in \beta$, then there is $B_3 \in \beta$ st $x \in B_3 \subset B_1 \cap B_2$

Then β is said to be a basis (of a topology) on X.

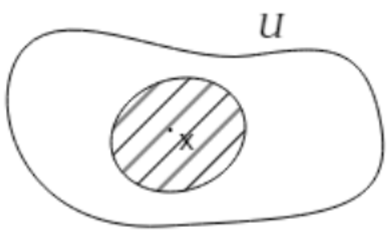
Definition 3.2.2. *If β is a basis as above, then the topology τ generated by β is defined as follows:*
$U \subset X$ is open in (X, τ) iff for any $x \in U$ there's $B \in \beta$ such that $x \in B \subset U$.
By definition, every basis element $B \in \beta$ is open in the topology generated by β.
$\tau = \{U \mid \forall x \in U \quad \exists B \in \beta \quad s.t. \quad x \in B \subset U\}$

Example 3.2.3. *Let β be the set of all circular regions in $\mathbb{R}^2$. Then β is a basis on $\mathbb{R}^2$.*

$B_3 \subset B_1 \cap B_2$

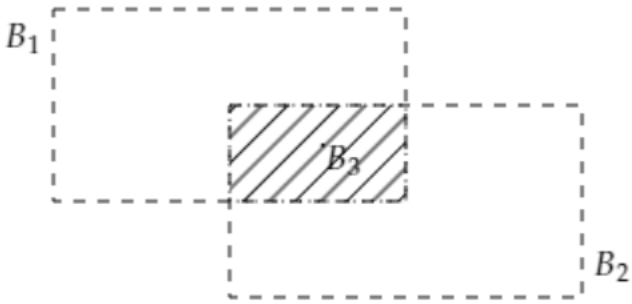


$(a, b) \in \mathbb{R}^2 \qquad\qquad (a, b) \in B_\epsilon = \{(x, y) \mid (x - a)^2 + (y - b)^2 < \epsilon\}$



U is open iff for every $x \in U$ there's a circular region around x that lies completely in U.

Example 3.2.4. *On $\mathbb{R}^2$, let β' be the set of all rectangular regions (excluding the bounding rectangle)*



Note that the intersection of two basis elements is a basis element itself: $B_3 = B_1 \cap B_2$. β' *is a basis for* $\mathbb{R}^2$.

Next, we prove that the topology generated by a basis in Definition 3.2.2 is a topology.

Theorem 3.2.5. *Let* β *be a basis on* X*, then* $\tau = \{U \mid \forall x \in U \quad \exists B \in \beta \quad st \quad x \in B \subset U\}$ *is a topology on* X.

Proof. 1) ϕ and $X \in \tau$

2) Let $\{U_\alpha\} \in \tau$ for $\alpha \in \wedge$

For $x \in \bigcup_{\alpha \in \wedge} U_\alpha$ then $x \in U'_\alpha$ for some $\alpha' \in \wedge$

This means there's $B' \in \beta$ st $X \in B' \subset U'_\alpha \subset \bigcup_{\alpha \in \wedge} U_\alpha$

$\Rightarrow \bigcup_{\alpha \in \wedge} U_\alpha \in \tau$

3) Let $U_1, U_2 \in \tau$ and $x \in U_1 \cap U_2 \Rightarrow x \in U_1$ and $x \in U_2$

$\Rightarrow$ there is $B_1, B_2 \in \beta$ st $x \in B_1 \subset U_1$ and $x \in B_2 \subset U_2$

So $x \in B_1 \cap B_2$. Since β is a basis there is $B_3 \in \beta$ st $x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2$. The rest follows by induction.

$\Rightarrow U_1 \cap U_2 \in \tau$

$\square$

Next, we see another description of open sets in a topology generated by a basis.

Theorem 3.2.6. *Let* X *be a set,* β *be a bais and* τ *be the topology generated by* β*. The family of unions of elements of* β *equals* τ.

Proof. Since $\beta \subset \tau$ any union of elements of β is in τ

Let $U \in \tau$ For any $x \in U$ there's $B_x \in \beta$ st $x \in B_x \subset U$

But $U = \bigcup_{x \in U} B_x$ $\square$

Remark 3.2.7. *Given a topology τ on X, can we construct a basis which generates the same topology as τ? A basis for a topology gives a shorter list of open sets and hence a simpler understanding of the properties of that space. A basis for a topology τ can be constructed as follows:*

Lemma 3.2.8. *Let (X, τ) be a topological space and $\mathcal{C}$ be a family of open sets in X such that for every open set $U \in \tau$ and every point $x \in U$ there is $C \in \mathcal{C}$ satisfying $x \in C \subset U$. Then $\mathcal{C}$ is a basis for τ.*

Example 3.2.9. *For any X, the family $\mathcal{C} = \{\{x\} | x \in X\}$ is a basis for the discrete topology on X.*

Theorem 3.2.10. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases for τ_1 and τ_2 respectively. Then, $\tau_1 \subset \tau_2$ if and only if for every $x \in X$ and $B_1 \in \mathcal{B}_1$ there is $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subset B_1$.*

Proof. ($\Rightarrow$) Let $U \in \tau_1$ (must show $U \in \tau_2$). Then there is $B_1 \in \mathcal{B}_1$ such that $x \in B_1 \subset U$ for all $x \in U$. By assumption there is $B_2 \in \mathcal{B}_2$ with $x \in B_2 \subset B_1 \subset U$.
($\Leftarrow$) Let $x \in X$ and $B_1 \in \mathcal{B}_1$ with $x \in B_1$. Since B_1 is open in τ_1 and since $\tau_1 \subset \tau_2$, we have $B_1 \subset \tau_2$. Since $\mathcal{B}_2$ generates τ_2 and $B_1 \in \tau_2$ there is $B_2 \in \mathcal{B}_2$ such that $B_2 \subset B_1$.

□

3.3 Topologies on $\mathbb{R}$

Definition 3.3.1. *(Standard Topology on $\mathbb{R}$) The family of open intervals $(a, b) = \{x \mid a < x < b\}$ is a basis $\mathcal{B}_{st}$ on $\mathbb{R}$ and topology it generates $\mathbb{R}_{st}$ is called the standard (or usual) topology on $\mathbb{R}$.*

$$\mathbb{R} = \bigcup_{n \in \mathbb{N}} (-n, n)$$

Definition 3.3.2. *Lower Limit Topology on $\mathbb{R}$. The family of intervals $[a, b) = \{x | a \leqslant x < b\}$ is a basis $\mathcal{B}_l$ and it generates $\mathbb{R}_l$.*

$$\mathbb{R} = \bigcup_{x \in \mathbb{R}} [x, y)$$

Theorem 3.3.3.

$$\mathbb{R}_{st} \subset \mathbb{R}_l$$

Proof. Let $x \in (a, b)$. Then $x \in [x, b) \subset (a, b)$ and $[x, b) \in \mathcal{B}_l$ and by Theorem 3.2.10 $\mathbb{R}_{st} \subset \mathbb{R}_l$.

2^{nd} **Way** : For $(a, b) \subset_{open} \mathbb{R}_{st}$
$x \in (a, b) = \bigcup_{n=1}^{\infty} [a + \frac{b-a}{2n}, b)$
Union of base elements in $\mathcal{B}$. So it must be open in $\mathbb{R}_l$. $\square$

Example 3.3.4. *Let* $K = \{\frac{1}{n} | n = 1, 2, 3...\}$ *and* $\mathcal{B}_K = \mathcal{B}_{st} \cup \{(a, b) - K | a < b\}$*. Denote the topology generated by* $\mathcal{B}_K$ *with* $\mathbb{R}_K$ *(K-topology)*

a. *Show* $\mathbb{R}_{st} \subset \mathbb{R}_K$
Since for every $x \in (a, b) \in \mathcal{B}_{st} \subset \mathcal{B}_K$ $(a, b) \in \mathcal{B}_K$*, this follows from Theorem 3.2.10.*
On the other hand, $B = (-1, 1) - K \in \mathcal{B}_K$
$0 \in B$ *and we can not find an open interval* (a, b) *such that* $0 \in (a, b) \subset B$*. Thus,* $\mathbb{R}_K \not\subset \mathbb{R}_{st}$

b. $\mathbb{R}_l$ *and* $\mathbb{R}_K$ *are not comparable.*
$[5, 7)$ *is open in* $\mathbb{R}_l$*, being a basis element.*
It is not open in $\mathbb{R}_K$ *since there is no basis element* $B \in \mathcal{B}_K$ *with* $5 \in B \subset [5, 7)$.
$(-1, 1) - K$ *is open in* $\mathbb{R}_K$. $0 \in (-1, 1) - K$ *but there is no* $[a, b)$ *with* $0 \in [a, b) \subset (-1, 1) - K$.

Definition 3.3.5. *A family of subsets of* X *is a subbase if the union of elements of this family is* X*. The topology generated by a subbase* S *consists of unions of finite intersections of elements of* S.

Chapter 4

Constructing Topologies

4.1 Order Topology

Let $<$ be an order relation on X, i.e.

i. For all $x, y \in X$ if $x \neq y$ then $x < y$ or $y < x$,

ii. There is no $x \in X$ with $x < x$,

iii. If $x < y$ and $y < z$ then $x < z$.

Define $(a, b) = \{x \mid a < x < b\}$, $[a, b) = (a, b) \cup \{a\}$, $(a, b] = (a, b) \cup \{b\}$, and $[a, b] = (a, b) \cup \{a, b\}$.

Definition 4.1.1. *Let $<$ be an order relation on X and suppose X has at least two elements. The family consisting of sets of the form,*

i. $(a, b) \subset X$ (open intervals)

ii. If there is a least element $a_0 \in X$, then intervals of type $[a_0, b)$

iii. If there is a greatest element $b_0 \in X$, then intervals of type $(a, b_0]$

is the basis of order topology on X.

Example 4.1.2. $\mathbb{R}_{st}$ *is also the order topology on* $\mathbb{R}$.

Example 4.1.3. *On $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ define the dictionary order topology with basis consisting of elements of the form,*
$(a \times b, c \times d) = \{x \times y \mid a \times b < x \times y < c \times d\}$
($a \times b < c \times d$ iff $a < c$ or " if $a = c$ then $b < d$")

Example 4.1.4. *Order topology on* $\mathbb{N} = \{1, 2, 3, ...\}$ *is equal to discrete topology*
$\{n\} = (n-1, n+1)$
$\{1\} = [1, 2)$
i.e. single element sets are basis elements, hence they are open.

Example 4.1.5. *Consider dictionary on* $X = \{1, 2\} \times \mathbb{N}$*.* X *has the least element* 1×1*. Denote the elements of the form* $1 \times n$ *and* $2 \times n$ *with* a_n *and* b_n*, respectively.*
$a_1 < a_2 < ...... < b_1 < b_2 < ......$
Question: *Is this also discrete topology?*
Answer: *No*
$\{a_1\} = [a_1, a_2)$
$\{a_n\} = (a_{n-1}, a_{n+1})$
$\{b_2\} = (b_1, b_3)$
$\{b_1\}$ *is not a basis element.*

4.2 Product Topology on $X \times Y$

In this section we define a topology on the cartesian product $X \times Y = \{(x, y) | x \in X, y \in Y\}$ of two topological spaces X and Y.

Definition 4.2.1. *Let* X *and* Y *be topological spaces. The family of sets of the form,*

$$U \times V \subset X \times Y$$

where $U \underset{open}{\subset} X$, $V \underset{open}{\subset} Y$ *is a basis* β *for the product topology on* $X \times Y$*.*

$$\beta = \{U \times V \mid U \underset{open}{\subset} X, V \underset{open}{\subset} Y\}$$

Question. *Is* β *a basis ?*

1) $X \subset_{open} X$, $\quad Y \subset_{open} Y$ *then* $X \times Y \in \beta$

2) *Let* $U_1 \times V_1$ *and* $U_2 \times V_2$ *be in* β*. Then* $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$

   $U_i \times V_i \in \beta \Rightarrow U_i \subset_{open} X \quad V_i \subset_{open} Y \quad i = 1, 2$

   $\Rightarrow U_1 \cap U_2 \subset_{open} X$ *and* $V_1 \cap V_2 \subset_{open} Y$

   $\Rightarrow U \times V \in \beta$

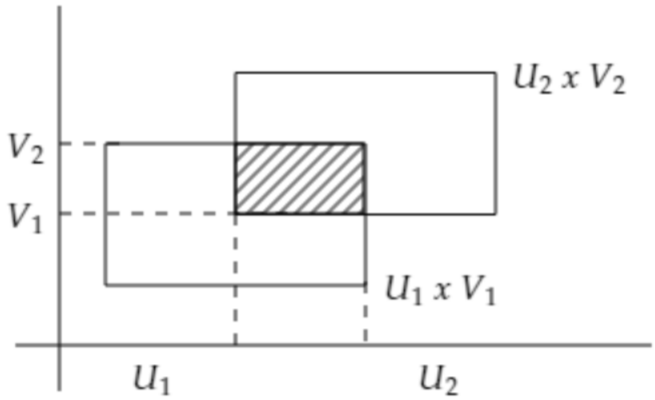
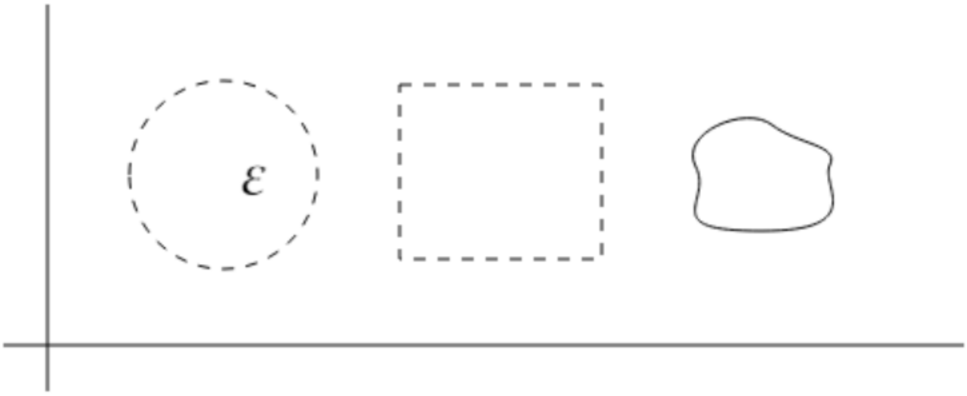


Figure 4.1: $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$

Theorem 4.2.2. *Let β_x and β_y be bases for the topological spaces X and Y resp.*

$D = \{A \times C \mid A \in \beta_x \ , \ C \in \beta_y\}$ *is a basis for the product topology on* $X \times Y$.

Example 4.2.3. $R_{st} \times R_{st} = R^2_{st}$ *standard topology on* R^2



By the above thm open rectangles form a basis for R^2_{st}

Definition 4.2.4.
$$\begin{array}{ll} \pi_1 : X \times Y \to X & \qquad \pi_2 : X \times Y \to Y \\ \quad (x,y) \to x & \qquad \quad (x,y) \to y \end{array}$$

are called projections onto X and Y.

* π_1 *and* π_2 *are onto*

* $U \subset_{open} X$ *then* $\pi_1^{-1}(U) = U \times Y \subset_{open} X \times Y$
$V \subset_{open} Y$ *then* $\pi_2^{-1}(V) = X \times V \subset_{open} X \times Y$

Theorem 4.2.5. $S_1 = \{\pi_1^{-1}(U) \mid U \subset_{open} X\}$ *and* $S_2 = \{\pi_2^{-1}(V) \mid V \subset_{open} Y\}$
Then $S = S_1 \cup S_2$ *is a subbase for the product topology on* $X \times Y$

Proof. Let τ : product topology on $X \times Y$
$\tau^{'}$: topology that S generates

Since $S \subset \tau$ and finite intersections of open sets and their unions is open the topology generated by S is in τ. (i.e. $\tau \subset \tau^{'}$)

On the other side, since basis elements of τ are
$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V) \in S \qquad \Rightarrow \qquad U \times V \in \tau^{'}$

□

4.3 Subspace Topology

Let (X, τ) be a topological space and $Y \subset X$ be a subset.
Then $\tau_Y = \{Y \cap U \mid U \in \tau\}$ is a topology on Y called the subspace topology on Y inherited from X.

* $\emptyset = Y \cap \emptyset$ and $Y = Y \cap X \Rightarrow \emptyset,\ Y \in \tau_y$
* $(U_1 \cap Y) \cap (U_2 \cap Y) \cap ... \cap (U_n \cap Y) = (U_1 \cap U_2 \cap ... \cap U_n) \cap Y$ (intersection of finitely many open sets)
* $\bigcup_{\alpha \in J}(U_\alpha \cap Y) = (\bigcup_{\alpha \in J} U_\alpha) \cap Y \in \tau_y$ (union of any number of open sets)

Theorem 4.3.1. *If* β *is a basis for* (X, τ) *then* $\beta_Y = \{B \cap Y \mid B \in \beta\}$ *is a basis for* (Y, τ_Y)

Definition 4.3.2. U *is said to be open* Y *if* $U \in \tau_Y$

Theorem 4.3.3. *If* Y *is open in* X *and* U *is open in* Y*, then* U *is open in* X*.*
$(U \subset_{open} Y, Y \subset_{open} X \Rightarrow U \subset_{open} X)$

Proof. If $U \subset_{open} Y$ then there's $V \subset_{open} X$ s.t. $U = Y \cap V$: intersection of two open sets in X
$\Rightarrow U \subset_{open} X$ □

Theorem 4.3.4. *Let* A *and* B *subspaces of* X *and* Y *respectively. The product topology on* $A \times B$ *is equal to subspace topology from* $X \times Y$

Proof. Let's show that these topologies have same bases.

Let $U \subset_{open} X$ and $V \subset_{open} Y$ be bases elements of X and Y resp.

Then $U \times V$ is a basis element for $X \times Y$. Thus $(U \times V) \cap (A \times B)$ is a basis element for the subspace topology on $A \times B$. Since $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$ which is a typical element of basis for the product topology on $A \times B$. See figure 4.1.

□

Example 4.3.5. $X = \{a, b, c, d, e\}$

$\tau = \{\emptyset, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$

$Y = \{a, d, e\}$

$\tau_Y = \{\emptyset, Y, \{a\}, \{d\}, \{a, d\}, \{d, e\}\}$

Example 4.3.6. $X = R_{st} \qquad Y = [0, 1]$

$\beta_y = \{(a, b) \cap [0, 1] \mid a, b \in \mathbb{R}\}$

$$(a, b) \cap [0, 1] = \begin{cases} (a, b) & a, b \in Y \\ [0, b) & b \in Y \quad a \notin Y \\ (a, 1] & a \in Y \quad b \notin Y \\ [0, 1] \ or \ \emptyset & a \notin Y and b \ \notin Y \end{cases}$$

Such elements form a basis for both subspace topology and order topology on Y

We see that $Y = [0, 1], (x, 1], [0, x]$ are open in Y.

Example 4.3.7. $X = R_{st} \qquad Y = [0, 1) \cup \{2\}$

$\{2\} = Y \cap (1, 3)$

$\Rightarrow \{2\}$ *is open in Y but not in X*

Note that $\{2\}$ is not open in Y if we have order topology on Y. Any base element containing 2 is of the form

$\{x \mid x \in Y \quad a < x \leq 2\}$ *for $a \in Y$. All those base elements must include elements that are less than 2.*

Chapter 5

Closed Sets and Limit Points

5.1 Closed Sets

Definition 5.1.1. *$A \subset X$ is a closed set if $X - A$ is open.*

Example 5.1.2. *$X = \mathbb{R}_{st}$.*
$[a, b]$ is closed since $\mathbb{R} - [a, b] = (-\infty, a) \cup (b, \infty)$ is open. $[a, \infty)$ and $(-\infty, b]$ are closed.

Example 5.1.3. *On a discrete topological space since every subset is open, every subset must be closed (being the complement of some open set).*

Example 5.1.4. *On a cofinite topological space finite sets are closed (together with $\emptyset$, X)*

Example 5.1.5. *On $\mathbb{R}^2_{st}$, let $Y = \{(x, y) \mid x \geq 0, y \geq 0\}$.*
$\mathbb{R}^2 \setminus Y = ((-\infty, 0) \times \mathbb{R}) \cup (\mathbb{R} \times (-\infty, 0))$ is open. So Y is closed.

Definition 5.1.6. *A is closed in the subspace Y if $Y - A$ is open in Y.*

Example 5.1.7. *$X = \mathbb{R}_{st}$ and let $Y = [0, 1] \cup (2, 3)$. Consider the subspace topology on Y,*
$[0, 1] = Y \cap (-1, 3/2)$ is open in Y.
$Y - [0, 1] = (2, 3)$ is closed in Y.
$(2, 3) = Y \cap (2, 3)$ is open in Y.

We can describe a topology using closed sets as well:

Theorem 5.1.8. *If X is a topological space,then*

i) $\emptyset, X$ are closed.

ii) Any intersection of closed sets is closed.

iii) Finite union of closed sets is closed.

Proof. ii) Let $\{A_\alpha\}_{\alpha\in J}$ be a family of closed sets.

$$X - \underset{\alpha\in J}{\cap} A_\alpha = \underset{\alpha\in J}{\cup}(X - A_\alpha)$$

is open (any union of open sets is open).

iii) $\{A_i\}_{i=1}^n$ be a finite collection of closed sets.

$$X - \overset{n}{\underset{i=1}{\cup}} A_i = \overset{n}{\underset{i=1}{\cap}}(X - A_i)$$

is open since finite intersection of open sets is open.

□

Theorem 5.1.9. *Let Y be a subspace of X. A is closed in Y iff $A = Y \cap C$ for some $C \underset{closed}{\subset} X$.*

Proof. ($\Leftarrow$) Let $C \underset{closed}{\subset} X$ and $A = C \cap Y$. Then $X - C \underset{open}{\subset} X$ and $(X - C) \cap Y \underset{open}{\subset} Y$.

$Y - A = (X - C) \cap Y$, A is closed in Y.

($\Rightarrow$) Let $A \underset{closed}{\subset} Y$ the $Y - A \underset{open}{\subset} Y \Rightarrow Y - A = Y \cap U$ for some $U \underset{open}{\subset} X$. $A = Y \cap (X - U)$ so if we let $C = X - U$ then C is closed in X and $A = Y \cap C$. □

Homework:$A \underset{closed}{\subset} Y$, $Y \underset{closed}{\subset} X$. Show that $A \underset{closed}{\subset} X$

5.1.1 Interior and Closure of a Set

Let X be a topological space and $A \subset X$. Interior of A is the union of all open sets that A covers and is denoted by A° .

$$A^{\circ} = \bigcup\{\ U \mid U \textit{ is open in } X \textit{ and } U \subset A\}$$

Being a union of open sets, A° is open. It is the largest open set that is included in A.

The closure of A is the intersection of all closed sets that cover A, denoted by $\overline{A}$.

$$\overline{A} = \bigcap\{\ K \mid K \textit{ is closed in } X \textit{ and } A \subset K\}$$

Being the intersection of closed sets $\overline{A}$ is closed. $\overline{A}$ is the smallest closed set that covers A.

Observe: If A is closed,then $\overline{A} = A$. If A is open, then $A^{\circ} = A$.

$$A^{\circ} \subset A \subset \overline{A}$$

Theorem 5.1.10. *Let X be a topological space, Y a subspace and $A \subset Y$. The closure of A in Y equals $\overline{A} \cap Y$.*

Proof. Let B denote the closure of A in Y. Since $\overline{A} \underset{closed}{\subset} X$ then $(\overline{A} \cap Y) \underset{closed}{\subset} Y$. But $A \subset (\overline{A} \cap Y)$ and hence $B \subset (\overline{A} \cap Y)$.On the other hand, since B is closed in Y, there's $C \underset{closed}{\subset} X$ such that $B = C \cap Y$.

$A \subset B = C \cap Y$ implies C is a closed set and $A \subset C$. $\Rightarrow \overline{A} \subset C \Rightarrow (\overline{A} \cap Y) \subset (C \cap Y) = B$ □

Theorem 5.1.11. *Let X be a topological space and $A \subset X$;*

a) $x \in \overline{A}$ iff every open set containing x intersects A ($x \in U \underset{open}{\subset} X \Rightarrow U \cap A \neq \emptyset$).

b) $x \in \overline{A}$ iff every basis element containing x intersects A.

Example 5.1.12. $X = \mathbb{R}_{st}$, $A = (0, 1]$ *then* $A^{o} = (0, 1)$, $\overline{A} = [0, 1]$.

Example 5.1.13. $B = \{1/n | n \in \mathbb{N}\}$ *then* $B^{o} = \emptyset$, $\overline{B} = B \cup \{0\}$.

Example 5.1.14. $\overline{\mathbb{Q}} = \mathbb{R}$.

Example 5.1.15. $\overline{\mathbb{N}} = \mathbb{N}$.

Example 5.1.16. $C = \{0\} \cup (1, 2) \Rightarrow \overline{C} = \{0\} \cup [1, 2]$.

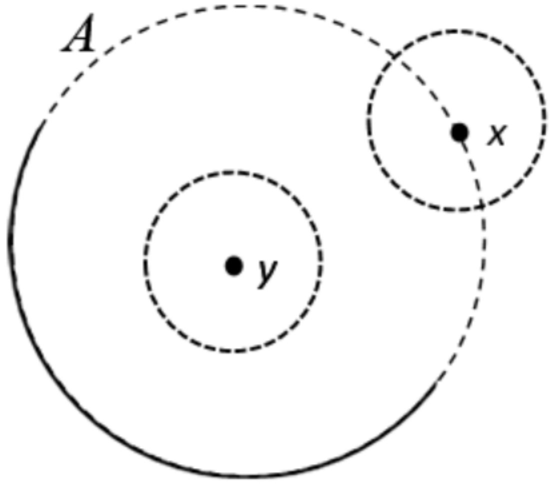
Example 5.1.17. $X = \mathbb{R}, Y = (0, 1]$
$A = (0, 1/2) \Rightarrow \overline{A} = [0, 1/2]$
$\Rightarrow \overline{A} \cap Y = (0, 1/2]$ *is the closure of A in Y.*

5.2 Limit Points

Let A be a subset of the topological space X and $x \in X$. Then x is a **limit point** of A, if every open set U containing x intersects A at some point other than x.
Equivalently, x is a limit point of A if it belongs to the closure of $A - \{x\}$.

Observe: x need not be in A to be a limit point of A.

Example 5.2.1. *x and y are limit points of A.*

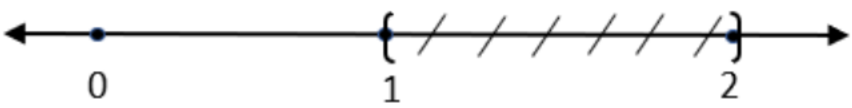


Definition 5.2.2. *An open set U containing $x \in X$ is said to be a* ***neighborhood*** *of x.*

Example 5.2.3. *$X = \mathbb{R}$ $A = (0, 1]$. Then any point $x \in [0, 1]$ is a limit point of A.*

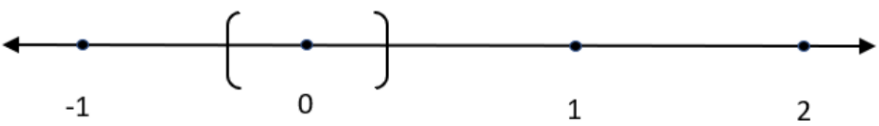
Example 5.2.4. *$B = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$. The only limit point of B is 0.*

Example 5.2.5. *$C = \{0\} \cup (1, 2)$. The set of limit points of C is $[1, 2]$.*



Example 5.2.6. *The set of limit points of $\mathbb{Q}$ in $\mathbb{R}$ is $\mathbb{R}$.*

Example 5.2.7. *$\mathbb{Z}$ has no limit points.*



Notation: A' denotes the set of limit points of A.

Theorem 5.2.8. $\overline{A} = A \cup A'$

Corollary 5.2.9. *A is closed if and only if $A' \subset A$.*

Definition 5.2.10. *The **boundary** of $A \subset X$ is $\partial A = \overline{A} \cap (\overline{X - A})$.*

Theorem 5.2.11. *i) ∂A and A° are disjoint.*

ii) $\overline{A} = A^\circ \cup \partial A$

iii) $\partial A = \emptyset$ iff A is both open and closed.

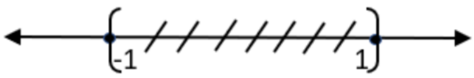
iv) A is open if and only if $\partial A = \overline{A} - A$

Proof. i) Let $x \in A^\circ$ so that x has a neighborhood U contained in A. For any $y \in U$ we have $y \in U \subset A$ hence $y \notin X - A$. $\Rightarrow U \cap (X - A) = \varnothing$.
$\star$(Recall that $x \in \overline{K}$ if and only if any neighbor of x intersects K.)
$\Rightarrow x \notin (\overline{X - A}) \Rightarrow x \notin \partial A = \overline{A} \cap (\overline{X - A})$

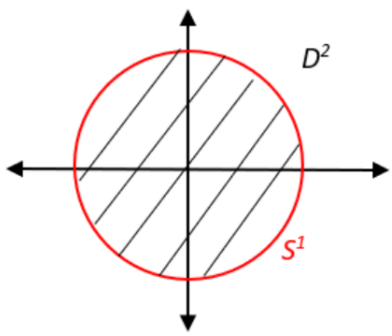
ii) $\subseteq$: Let $x \in \overline{A}$. If $x \in A^\circ$ then $x \in A^\circ \cup \partial A$. Suppose $x \notin A^\circ$ (must show: $x \in \partial A = \overline{A} \cap (\overline{X - A})$). Then for any neighborhood U of x, $U \cap (X - A) \neq \varnothing$. By $\star$, $x \in (\overline{X - A})$.
$\supseteq$: Let $x \in A^\circ \cup \partial A$. If $x \in A^\circ$ then $x \in A^\circ \subset A \subset \overline{A}$. If $x \in \partial A = \overline{A} \cap (\overline{X - A})$, then $x \in \overline{A}$.

$\square$

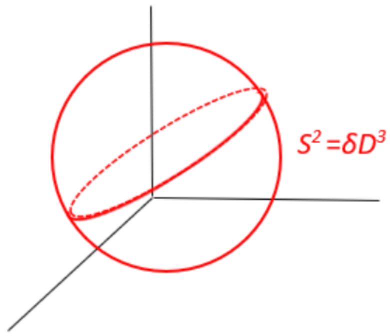
Example 5.2.12. *Let $D^n = B^n = \{(x_1, x_2, ..., x_n) \mid x_1^2 + x_2^2 + ... + x_n^2 \leq 1\}$ denote the unit n-disk or the n-ball which is a subset of $\mathbb{R}^n$.*
The boundary of D^n: $\partial D^n = S^{n-1} = \{(x_1, x_2, ..., x_n) \mid x_1^2 + x_2^2 + ... + x_n^2 = 1\}$ is called the (n-1)-sphere.
The interior of D^n : $(D^n)^\circ = \{(x_1, x_2, ..., x_n) \mid x_1^2 + x_2^2 + ... + x_n^2 < 1\}$ is the open n-disk.
<u>n=1:</u> $D^1 = [-1, 1] = \{x \mid x^2 \leq 1\}$ $\partial D^1 = S^0 = \{-1, 1\}$.



<u>n=2:</u> $D^2 = \{(x, y) \mid x^2 + y^2 \leq 1\}$ $\partial D^2 = S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$*: unit circle.*



$\underline{n=3}$: $D^3 = \{(x,y,z) \mid x^2+y^2+z^2 \leq 1\}$ $\quad \partial D^3 = S^2 =$ *unit sphere.*



Example 5.2.13. $\partial\mathbb{R} = \emptyset = \overline{\mathbb{R}} \cap \overline{(\mathbb{R}-\mathbb{R})} = \mathbb{R} \cap \emptyset$

Example 5.2.14. $\partial\mathbb{Q} = \overline{\mathbb{Q}} \cap \overline{(\mathbb{R}-\mathbb{Q})} = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$
OR $\partial\mathbb{Q} = \overline{\mathbb{Q}} - \mathbb{Q}^\circ = \mathbb{R} - \emptyset = \mathbb{R}$

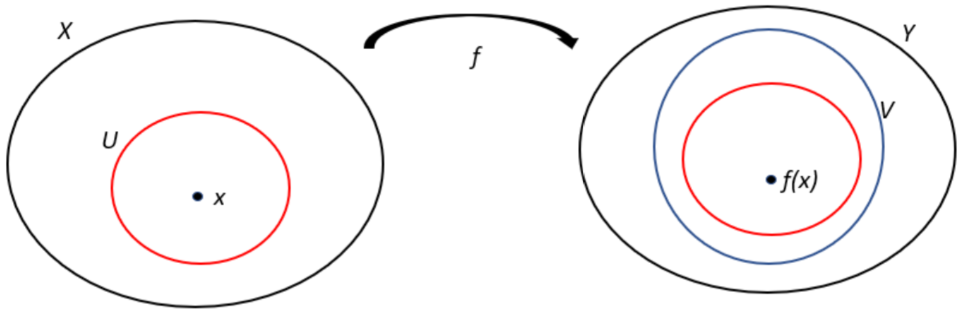
$\diamond$ The boundary of A is also defined as the set of boundary points. A point $p \in X$ is a boundary point of A if every neighborhood of p contains at least one point of A and at least one point not of A.

Chapter 6

Continuous Functions

Let X and Y be topological spaces. A function $f : X \to Y$ is **continuous** if the inverse image of every open set in Y is open in X i.e. "$V \subset_{open} Y \Rightarrow f^{-1}(V) \subset_{open} X$ if and only if f is continuous."

Pointwise : f is continuous at $x \in X$ if and only if for any neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.



Remark: $\varepsilon - \delta$ definition of Calculus 1 is equivalent to this definition.
"$f : \mathbb{R} \to \mathbb{R}$ is continuous at $x \in \mathbb{R} \Leftrightarrow \forall \varepsilon > 0 \quad \exists \delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$."
For every neighborhood of $f(x)$ $(V = (f(x) - \varepsilon,\ f(x) + \varepsilon))$ there is a neighborhood of x $(U = (x - \delta,\ x + \delta))$ such that if $y \in (x - \delta, x + \delta)$ then $f(y) \in (f(x) - \varepsilon,\ f(x) + \varepsilon)$.
i.e. $f((x - \delta, x + \delta)) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$.

As usual, working with basis elements, rather than open sets is more convenient.

Theorem 6.0.1. *If $\mathcal{B}$ is a basis for the topology on Y and for every basis element $B \in \mathcal{B}$ if $f^{-1}(B)$ is open in X, then $f : X \to Y$ is continuous.*

Proof. $V \subset_{open} Y \Rightarrow V = \bigcup_{\alpha \in J} B_\alpha \Rightarrow f^{-1}(V) = \bigcup_{\alpha \in J} f^{-1}(B_\alpha) \subset_{open} X$. $\square$

Example 6.0.2. *Let $\mathbb{R}_{st}$ and $\mathbb{R}_l$ denote standard and lower limit topologies on $\mathbb{R}$, respectively. The identity function*

i) $\begin{array}{cccc} f: & \mathbb{R}_{st} & \to & \mathbb{R}_l \\ & x & \to & x \end{array}$ *is not continuous but*

ii) $\begin{array}{cccc} f: & \mathbb{R}_l & \to & \mathbb{R}_{st} \\ & x & \to & x \end{array}$ *is continuous.*

For

i) $[a, b) \underset{open}{\subset} \mathbb{R}_l$ *but* $f^{-1}([a, b)) = [a, b)$ *is not open in* $\mathbb{R}_{st}$.

ii) $(a, b) \underset{open}{\subset} \mathbb{R}_{st}$ *and* $f^{-1}((a, b)) = (a, b) \underset{open}{\subset} \mathbb{R}_l$.

Example 6.0.3. *Any function $f : X \to Y$ is continuous if X has discrete topology.*

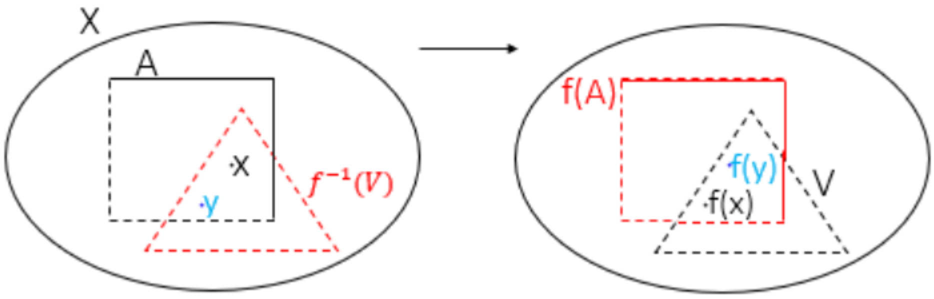
Example 6.0.4. *Any function $f : X \to Y$ is continuous if Y has trivial(indiscrete) topology.*

Theorem 6.0.5. *Let X and Y be topological spaces, and $f : X \to Y$ be a function.*
Then the following are equivalent:

1) *f is continuous.*
2) *For any subset A of X, we have $f(\overline{A}) \subset \overline{f(A)}$.*
3) *For every closed set B of Y, $f^{-1}(B)$ is closed in X.*
4) *For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) and (1) $\Rightarrow$ (4) $\Rightarrow$ (1)

(1) $\Rightarrow$ (2): Let f be continuous, A be a subset of X and $x \in \overline{A}$.
(must show: $f(x) \in \overline{f(A)}$, i.e $f(x)$ is a limit point of$f(A)$)
Let V be a neighborhood of $f(x)$. Then $f^{-1}(V) \underset{open}{\subset} X$ must be a neighborhood of x and hence it must intersect A, (because x is a limit point of A), at some point y. Then V intersect $f(A)$ in the point $f(y)$, so that $f(x)$ is a limit point of $f(A)$.
$\Rightarrow f(x) \in f(A)$

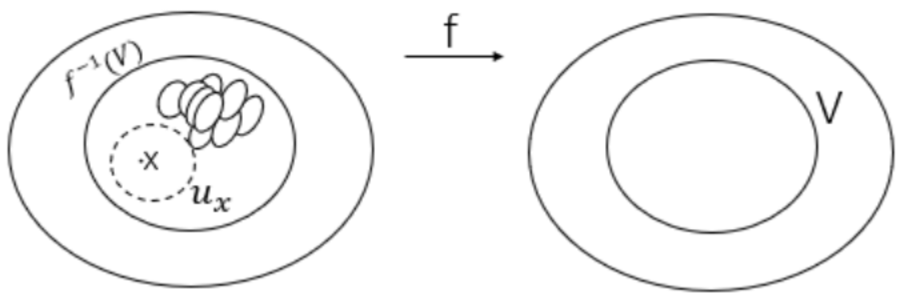


(2) $\Rightarrow$ (3): Let B be closed in Y, and set $A = f^{-1}(B)$. (must show: $\overline{A} = A$)
$f(A) = f(f^{-1}(B))) \subset B$ (Worksheet-1)
Thus, if $x \in \overline{A}$, then $f(x) \in f(\overline{A}) \subset \overline{f(A)} \subset \overline{B} = B$
This means $x \in f^{-1}(B) = A$ i.e $\overline{A} \subset A$.

(3) $\Rightarrow$ (1): For $V \underset{open}{\subset} Y$ set $B = Y - V$ which is closed.
Then $f^{-1}(B) = f^{-1}(Y - V) = f^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V)$ is closed by the assumption. Hence $f^{-1}(V) \underset{open}{\subset} X$.

(1) $\Rightarrow$ (4): Let $x \in X$ and V be a neighborhood of $f(x)$.
The set $U = f^{-1}(V)$ is a neighborhood of x such that $f(U) \subset V$.
$(f(U) = f(f^{-1}(V)) \subset V)$

(4) $\Rightarrow$ (1): Let V be open in Y and x be a point in $f^{-1}(V)$.
Then $f(x) \in V$. By the assumption, there is a neighborhood U_x of x such that $f(U_x) \subset V$. Then $U_x \subset f^{-1}(V)$. This means $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$ i.e union of open sets.
$\Rightarrow f^{-1}(V)$ is open.



$\square$

6.1 Homeomorphisms

Let X and Y be topological spaces, $f : X \to Y$ a bijection. If both f and f^{-1} are continuous then f is called a homeomorphism. In this case X and Y are said to be homeomorpic spaces.
Equivalently, a homeomorphism is a bijection $f : X \to Y$ such that "$f(U)$ is open if and only if U is open". This means: a homeomorphism is not only a $1-1$ correspondence between elements of X and Y, it is also a $1-1$ correspondence between open sets of X and Y. Any property of X defined through open sets (connected, compact, Haussdorff, ...) also holds for Y.

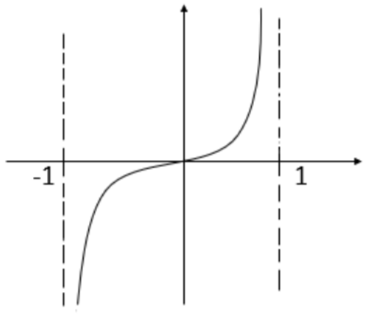
Example 6.1.1. *Let $X = \{x, y, z\}$ and $Y = \{1, 2, 3\}$, and $\tau_X = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$, $\tau_Y = \{\emptyset, X, \{2\}, \{3\}, \{2, 3\}\}$ be topologies of X and Y, respectively.*
Is X and Y homeomorphism?
***Yes**. Let $f : X \to Y$ be defined with $f(a) = 2$, $f(b) = 1$, $f(c) = 3$. ($1-1$ and onto)*
f takes open sets of X to open sets of Y.

$$\begin{array}{ccc} \emptyset & \leftrightarrow & \emptyset g \\ X & \leftrightarrow & X \\ \{a\} & \leftrightarrow & \{2\} \\ \{c\} & \leftrightarrow & \{3\} \\ \{a, c\} & \leftrightarrow & \{2, 3\} \end{array}$$

Example 6.1.2. *Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2 = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$: discrete topology. Is the identity function $(X, \tau_1) \to (X, \tau_2)$ a homeomorphism?*
***Answer:** $f : X \to X$ $f(x) = x$ is clearly bijection and its inverse is itself. But f is not continuous : $f^{-1}(\{c\}) = \{c\} \notin \tau_1 \Longrightarrow f$ is not a homeomorphism.*
***Show:** There is no such homeomorphism $(X, \tau_1) \to (X, \tau_2)$.*

Example 6.1.3. $\begin{array}{rcl} F : (-1, 1) & \to & \mathbb{R}_{st} \\ x & \mapsto & \frac{x}{1-x^2} \end{array}$ *(We consider $X = (-1, 1)$ with subspace topology from $\mathbb{R}_{st}$)*
$F' \geq 0 \Longrightarrow F$ is increasing $\Longrightarrow F$ is 1-1
$F^{-1} = G(y) = \frac{2y}{1+\sqrt{1+4y^2}}$ is defined for all $y \in \mathbb{R} \Longrightarrow F$: onto.
Both F and G are continuous $\Longrightarrow F$ is a homeomorphism.

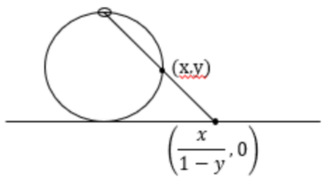
Example 6.1.4. *(Stereographic Projection)*



$S^n - \{1point\}$ *is homeomorphic to* $\mathbb{R}^n$

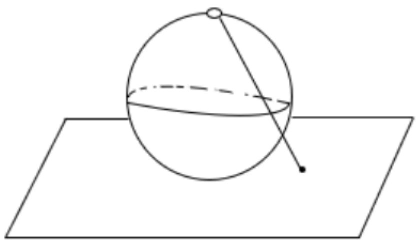
$n=1$: $\quad f: \begin{array}{ccc} S' & \to & \mathbb{R} \\ (x,y) & \mapsto & \frac{x}{1-y} \end{array}$

$f^{-1}: \begin{array}{ccc} \mathbb{R} & \to & S' \\ x & \mapsto & \left(\frac{2x}{1+x^2}, \frac{-1+x^2}{1+x^2}\right) \end{array}$ *are both continuous.*



$n=2$: $\quad f: \begin{array}{ccc} S^2 - \{northpole\} & \to & \mathbb{R}^2 \\ \{x,y,z\} & \mapsto & \left(\frac{x}{1-z}, \frac{y}{1-z}\right) \end{array}$

$f^{-1}(a,b) = \left(\frac{2a}{1+a^2+b^2}, \frac{2b}{1+a^2+b^2}, \frac{-1+a^2+b^2}{1+a^2+b^2})\right)$



The result of this example is sometimes expressed as: $\mathbb{R}^n \cup \{\infty\} = S^n$

Theorem 6.1.5. *Let X, Y, Z be topological spaces.*

i) Constant functions are continuous.

ii) $A \subset X$ $i : A \hookrightarrow X$ the inclusion function $i = id|_A$ is continuous.

iii) If $f : X \to Y$ and $g : Y \to Z$ are continuous, so is $g \circ f : X \to Z$.

iv) If $f : X \to Y$ is continuous and $A \subset X$, then $f|_A$ is continuous.

Proof. i) Let $f : X \to Y$ constant i.e. $f(x) = a \in Y$ for every $x \in X$. For any open set $V \subset Y$ $f^{-1}(V) = \emptyset$ if $a \notin V$
$f^{-1}(V) = X$ if $a \in V \implies f$ is continuous.

iii) $U \subset_{open} Z \implies (g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$
$((g^{-1}(U))$ is open since g is constant, $f^{-1}(g^{-1}(U))$ is open since f is continuous.)

ii) $i : A \to X$ let $U \subset_{open} X$. $i^{-1} = U \cap A$ ($U \cap A$ open in A with subspace topology)

iv) $f_{|A} = (f \circ i) : A \hookrightarrow X \to Y$ composition is continuous.

$\square$

Theorem 6.1.6. *(Pasting Lemma) Let A and B be closed in X s.t. $X = A \cup B$. If $f : A \to Y$ and $g : B \to Y$ are continuous and $f(x) = g(x)$ for any $x \in A \cap B$, then*

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases} \text{ is continuous } X \to Y$$

Example 6.1.7. $h(x) = \begin{cases} x & \text{if } x \leq 0 \\ \frac{x}{2} & \text{if } x > 0 \end{cases}$ *is continuous.*

Theorem 6.1.8. $\begin{array}{rccc} f: & A & \to & X \times Y \\ & a & \mapsto & (f_1(a), f_2(a)) \end{array}$ *is continuous iff f_1 and f_2 are continuous.*

Proof. ($\Longrightarrow$) $f_1 = \pi_1(f)$ and $f_2 = \pi_2(f)$ are continuous, where π_i is the projection onto i^{th} component.

($\Longleftarrow$) Recall that basis elements in $X \times Y$ are of the form $U \times V$, where U and V are basis element for X and Y, respectively.

$f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$ open. ($f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open since f_1 and f_2 are continuous.) $\square$

Chapter 7

Product and Metric Topologies

7.1 Two topologies on a Product Space

We have seen the product topology on $X \times Y$ in section 4.2. Let's generalize this to products of the form: $X = X_1 \times X_2 \times ... \times X_n = \prod_{i=1}^{n} X_i$ or
$Y = X_1 \times X_2 \times X_3 \times ... = \prod_{i=1}^{\infty} X_i$.

Definition 7.1.1. *(Box Topology) For open sets $U_i \subset X_i$ the sets of the form*

$$U_1 \times U_2 \times ... \times U_n \quad and$$
$$U_1 \times U_2 \times U_3 \times ...$$

are bases elements of the box topologies on X and Y above, respectively. i can be taken from any index set.

Definition 7.1.2. *If $\pi_i : X or\ Y \to X_i$ is the i^{th} projection then $\mathfrak{B} = \{\pi_i^{-1}(U_i) | U_i \subset_{open} X_i\}$ is a subbasis for the product topology on X or Y.*

Box topology and product topology are equal on the finitely many product of spaces i.e on $X = X_1 \times X_2 \times ... \times X_n$.

Theorem 7.1.3. *Let $f : A \to \prod_{\alpha \in j} X_\alpha$ be the function defined by $f(a) = (f_\alpha(a))_{\alpha \in j}$ for $f_\alpha : A \to X_\alpha$. f is continuous iff f_α is continuous for all $\alpha \in j$.*

In this theorem we assume $\Pi\ X_\alpha$ has product topology. Product topology is stronger than box topology on the product of infinitely many spaces.

Theorem 7.1.4. *(Comparison of the box and product topologies) The box topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α.*
The product topology $\prod X_\alpha$ has as basis of the form $\prod U_\alpha$ is open in X_α for each α, and U_α equals X_α except for finitely many valuse of α.

Example 7.1.5. *For* $X_n = \mathbb{R}$*, let* $\mathbb{R}^w = \prod_{n\in\mathbb{N}} X_n = \{(x_1, x_2, ..., x_n, ...) \mid x_n \in \mathbb{R}, n \in \mathbb{N}\}$

$$\text{Define } f : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R}^w \\ t & \mapsto & (t, t, t, ...) \end{array}$$

so that $f_n(t) = t$ *is the* n^{th} *coordinate function.* f *is continuous if* $\mathbb{R}^w$ *has product topology by theorem 7.1.3, but not if it has the box topology.*

Consider $B = (-1, 1) \times \left(\frac{-1}{2}, \frac{1}{2}\right) \times \left(\frac{-1}{3}, \frac{1}{3}\right) \times ...$ *which is a basis element of the box topology on* $\mathbb{R}^w$*.*

Claim. $f^{-1}(B)$ *is not open in* $\mathbb{R}$

Since $(0, 0, 0, ..) \in B$, $0 \in f^{-1}(B)$*. If* $f^{-1}(B)$ *is open then* 0 *must have a neighborhood* $(-\delta, \delta)$ *s.t.* $f((-\delta, \delta)) \subset B$*. But* $f((-\delta, \delta)) = (-\delta, \delta) \times (-\delta, \delta) \times ... \subset (-1, 1) \times \left(\frac{-1}{2}, \frac{1}{2}\right) \times ...$ *is impossible.*

7.2 Metric Topology

Remember that a metric d on a set X is a (distance) function $d : X \times X \to \mathbb{R}$ such that:

1) $d(x, y) \geq 0$ for all $y, x \in X$ and $d(x, y) = 0 \iff x = y$,

2) $d(x, y) = d(y, x)$ for all $x, y \in X$,

3) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y \in X$.

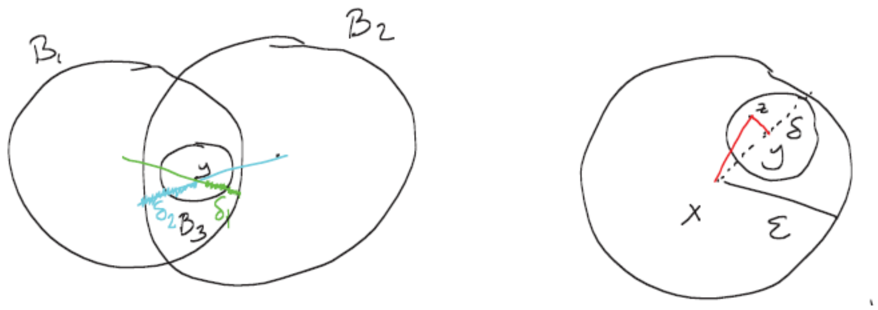
In this case, $B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$ is called the ϵ-ball centered at x.

Definition 7.2.1. *The collection of* ϵ*-balls* $B_d(x, \epsilon)$ *for* $x \in X$ *and* $\epsilon > 0$ *is a basis for a topology on* X*, called the metric topology induced by* d*.*

Example 7.2.2. *Is* $\mathbb{B} = \{B_d(x, \epsilon) | x \in X, \epsilon > 0\}$ *a basis ?*
1- $X = \bigcup\{B_d(x, \epsilon) \mid x \in X,\ \epsilon > 0\}$
2- Next, we need to show that for any couple of ϵ*-balls* B_1 *and* B_2*, and any* $y \in B_1 \cap B_2$*, there is a third basis element* B_3 *(another* ϵ*-ball) such that* $B_3 \subset B_1 \cap B_2$ *as in Figure* **??** *(a)*

Given an ϵ*-ball* $B(x, \epsilon)$ *and* $y \neq x$ *in this ball, we can find* $B(y, \delta) \subset B(x, \epsilon)$*, by simply letting* $\delta = \epsilon - d(x, y)$*. If* $z \in B(y, \delta)$*, then since* $d(x, z) \leq d(x, y) + d(y, z) \leq d(x, y) + \delta \leq \epsilon$*. This implies* $z \in B(x,\epsilon)$ *.(Figure* **??** *(b)) Then taking the smaller of such balls for points in the intersection, we can fulfill item 2 above.*

There are positive numbers δ_1 *and* δ_2 *such that* $B(y, \delta_1) \subset B_1$ *and* $B(y, \delta_2) \subset B_2$*. Now letting* $\delta = \min\{\delta_1, \delta_2\}$ *we get* $B_3 = B(y, \delta) \subset B_1 \cap B_2$*.*



(a) (b)

Definition 7.2.3. *A set U is open in this topology if for each $y \in U$ there is a $\delta > 0$ such that $B_d(y, \delta) \in U$. Obviously ϵ-balls are open.*

Example 7.2.4. *Discrete metric* $d(x,y) = \begin{cases} 1 & \text{for } x \neq y \\ 0 & \text{for } x = y \end{cases}$ *induces the discrete topology.* B_d*=(x,1)=*$\{x\}$

Example 7.2.5. *Standard metric on $\mathbb{R}$ given by $d(x,y)=|x-y|$ induces $\mathbb{R}_{st}$ with its order topology. If $x = (a+b)/2$ then $(a,b) = B(x, (b-a)/2)$ (i.e. open intervals are ϵ-balls) and $B(x, \epsilon) = (x - \epsilon, x + \epsilon)$ (i.e. ϵ-balls are open intervals).*

Definition 7.2.6. *If the topology on X is induced by a metric d, then (X, τ) is called metrizable.*

Theorem 7.2.7. *Let d_1 and d_2 induce τ_1 and τ_2 on X respectively. Then, $\tau_1 \subset \tau_2$ if and only if for every $x \in X$, and $\epsilon > 0$ there is $\delta > 0$ such that $B_{d_2}(x, \delta) \subset B_{d_1}(x, \epsilon)$.*

Proof. This is theorem 3.2.10 with metric spaces and ϵ-balls as their bases elements. $\square$

Theorem 7.2.8. *The topology on $\mathbb{R}^n$ induced by the euclidean metric $d(x,y)=\sqrt{(x_1-y_1)^2 + ... + (x_n - y_n)^2}$ and the square metric $\rho(x,y) = \max\{|x_1 - y_1|, ...|x_n - y_n|\}$ are the same as the product topology on $\mathbb{R}^n$.*

Proof. Observe that $\rho(x,y) \leq d(x,y) \leq \sqrt{n}\rho(x,y)$. The first inequality implies that $B_d(x, \epsilon) \subset B_\rho(x, \epsilon)$.
Suppose $y \in B_\rho(x, \frac{\epsilon}{\sqrt{n}}) \Rightarrow \rho(x,y) < \frac{\epsilon}{\sqrt{n}}$
$\Rightarrow d(x,y) < \sqrt{n} \cdot \frac{\epsilon}{\sqrt{n}} = \epsilon$
$\Rightarrow y \in B_d(x, \epsilon/)$ and hence $\Rightarrow B_\rho(x, \frac{\epsilon}{\sqrt{n}}) \subset B_d(x, \epsilon)$.

Thus, from the previous theorem $\tau_d \subset \tau_\rho \subset \tau_d$ i.e. the topologies are equal.

To show that they are both equal to product topology, let B=$((a_1, b_1)\times...(a_n, b_n))$ be a basis element and X=$(x_1, ...x_n) \in$ B for all i=1,2,...,n. There is ϵ_i s.t. $(x_i - \epsilon_i, x_i + \epsilon_i) \subset (a_i, b_i)$. Choose $\epsilon = \min\{\epsilon_1, ...\epsilon_n\}$, then $B_\rho(x, \epsilon) \subset$ B.
On the other hand, if y $\in B_\rho(x, \epsilon)$ then we need to find B such that y $\in B \subset B_\rho(s, \epsilon)$. But
$B_\rho(x, \epsilon) = \{y \in \mathbb{R}^n \mid max\,\{|x_1 - y_1|, ...|x_n - y_n|\} < \epsilon\}$
$= \{y \in \mathbb{R}^n \mid |x_1 - y_1| < \epsilon, ...|x_n - y_n| < \epsilon\}$
$= \{(y_1, ..., y_n) \in \mathbb{R}^n \mid y_1 \in (x_1 - \epsilon, x_1 + \epsilon), ..., y_n \in (x_n - \epsilon, x_n + \epsilon)\}$
$= (x_1 - \epsilon, x_1 + \epsilon) \times ... \times (x_n - \epsilon, x_n + \epsilon) =$ B. □

Definition 7.2.9. *Let* $\{x_n\} = \{x_1, x_2, ..., x_n, ... \mid x_n \in X\}$ *be a sequence in a topological space* X*. It converges to* $x \in X$ *if for every neighborhood* U *of* x*, there is* $\bar{n} \in \mathbb{N}$ *such that* $x_n \in U$*, whenever* $n \geq \bar{n}$*.*

Theorem 7.2.10. *Let* A *be a subset of a topological space* X *and* $\{a_n\} \subset A$ *be a sequence converging to* x*. Then* $x \in \bar{A}$*. The converse is true only if* X *is metrizable.*

Proof. Recall that $x \in \bar{A}$ iff any neighborhood of x intersects A. Since $lima_n = x$, any neigborhood of x contains infinitely many terms of $\{a_n\} \subset A$.
For the converse, see worksheet 1, Q9. □

Theorem 7.2.11. *If* $f : X \to Y$ *is continuous and* $\lim x_n = x$*, then* $\lim f(x_n) = f(x)$*. The converse is true if* X *is metrizable.*

Proof. Let $V \subset Y$ be a neighborhood of $f(x)$. Since f is continuous, $f^{-1}(V)$ is a neighborhood of x. Therefore there is $\bar{n} \in \mathbb{N}$ such that $n \geq \bar{n}$ implies $x_n \in f^{-1}(V)$ or $\{\lambda_n\}_n^\infty \subset f^{-1}(V)$.
On the other hand, let $A \subset X$ and $x \in \bar{A}$. By the above theorem, there is a sequence $\{x_n\} \subset A$ such that $\lim x_n = x$. Since $\lim f(x_n) = f(x)$ and $f(x_n) \subset f(A)$ we have $f(x) \in \overline{f(A)}$. This gives $f(\bar{A}) \subset \overline{f(A)}$ and hence f is continuous. □

Definition 7.2.12. *Let* $f_n : X \to Y$ *be sequence of functions from the set* X *to the metric space* (Y, d)*.* $\{f_n\}$ *is said to converge uniformly to* $f : X \to Y$*, if given* $\epsilon > 0$ *there is an integer* $\bar{n} \in \mathbb{N}$ *such that* $d(f_n(x), f(x)) < \epsilon$ *for all* $n > \bar{n}$ *and* $x \in X$*.*

Theorem 7.2.13. *If* $\{f_n\}$ *and* f *are as in the above definition, then* f *is continuous.*

Chapter 8

Connected Spaces

The intermediate value theorem we have seen in Calculus courses relies on a property that the closed interval $[a, b]$ has.

Theorem 8.0.1 (Intermediate Value Theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous and $r \in \mathbb{R}$ is between $f(a)$ and $f(b)$, then there is $c \in [a, b]$ such that $f(c) = r$.*

The property of $[a, b]$ that makes IVT true is connectedness. Since f is continuous, we will see that $f([a, b])$ is also connected.

Definition 8.0.2. *Let X be topological space. A separation of X is a pair U, V of disjoint nonempty open subsets of X, whose union is X.*
If X has no separation, then X is called connected.

Connectedness is a topological property, (A property is topological if it is preserved under a homeomorphism) since it is defined through open sets. If U, V is a separation of X, then $U = X - V$ and $V = X - U$. This means U and V are both open and closed. In other words, X is connected if and only if $\emptyset$ and X are the only sets that are both open and closed.

Theorem 8.0.3. *Let Y be a subspace of X and A and B be disjoint nonempty sets of Y, with $Y = A \cup B$. The pair A, B is a separation for Y (i.e. Y is disconnected) if and only if neither A nor B contains a limit point of the other.*

Proof. ($\Rightarrow$) (Recall that $\overline{K} = K \cup K'$). We need to show that closures of A and B in Y are also disjoint. Since A and B form a separation, A and B are both open and closed in Y. The closure of A in Y is $\overline{A} \cap Y$. Since A is closed in Y we have $A = \overline{A} \cap Y$ (similarly $B = \overline{B} \cap Y$ is the closure of B in Y). Therefore closures of A and B are disjoint in Y.

($\Leftarrow$) Suppose A and B are disjoint nonempty sets whose union is Y, neither of which contains a limit point of the other. Then $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$. To show that A is closed in Y, note that $\overline{A} \cap (Y - A) = (\overline{A} \cap Y) - (\overline{A} \cap A) = (\overline{A} \cap Y) - A = \emptyset \Rightarrow A = \overline{A} \cap Y$. i.e. all limit points of A are in A, implying that A is closed.
Similarly $B = \overline{B} \cap Y$. A and B are both closed and open. □

Example 8.0.4. *1) $X = \{a, b\}$ with the trivial topology $\tau = \{\emptyset, X\}$.*
$\Rightarrow X$ is connected

2) Y= $[-1, 0) \cup (0, 1] = [-1, 1]\backslash\{0\} \subset \mathbb{R}$ is disconnected.
$[-1, 0)$and $(0, 1]$are both open and closed in Y, hence they form a separation.

3) $\mathbb{Q}$ is not connected. Only connected sbspaces of $\mathbb{Q}$ are the one point sets If $Y \subset \mathbb{Q}$ has two points p and q, then one can choose an irrational number a between p and q and write Y as
Y=$[Y \cap (\infty, a)] \cup [Y \cap (a, \infty)]$ union of disjoint sets.

Example 8.0.5. *$X = \{(x, y) \mid y = 0\} \cup \{(x, y) \mid x > 0$ and $y = 1/x\}$ is a subset of $\mathbb{R}^2$ which is not connected.*
Neither of the two subsets of X above contain limit points of the other, hence they form a separation.

8.1 Constructing Connected Spaces

Lemma 8.1.1. *If the sets C and D form a separation of X and Y is a connected subspace of X, then Y lies entirely within C or D.*

Proof. C and D are both open and closed in X. The sets $C \cap Y$ and $D \cap Y$ are open in Y and
$(C \cap Y) \cap (D \cap Y)) = (C \cap D) \cap Y = \emptyset \cap Y = \emptyset)$
$(C \cap Y) \cup (D \cap Y) = Y \cap (C \cup D) = Y \cap X = Y$.
i.e. they form a separation for Y if both of them are nonempty. Since Y is connected one of them must be empty. □

Theorem 8.1.2. *The union of a collection of connected subspaces of X that have a point in common is connected.*

Proof. Let $\{A_\alpha\}$be a collection of connected subspaces of X, and $p \in \bigcap A_\alpha$
<u>*Claim*</u> : $Y = \bigcup A_\alpha$ is connected.
Suppose not and assume $Y = C \cup D$ is a separation of Y. By previous lemma, p is in one of C or D. Let $p \in C$. Since A_α is connected for each α, it must

lie in C or D and cannot lie in D because $p \in C$. Hence $A \subset C$ for all α and $\bigcup A_\alpha \subset C$ contradicting the fact that $D \neq \emptyset$ $\square$

Theorem 8.1.3. *Let A be a connected subspace of X. If $A \subset B \subset \overline{A}$, then B is also connected.*

The theorem says if we add to a connected set some or all of its limit points, then we still preserve connectedness.

Proof. Let A be connected and $A \subset B \subset \overline{A}$. Suppose B is not connected and $B = C \cup D$ is a seperation of B. By above lemma, A must lie entirely in C or D. WLOG let $A \subset C$. Then $\overline{A} \subset \overline{C} = C$. Since C and D are disjoint and $A \subset B \subset \overline{A} \subset \overline{C} = C$ we have $B \cap D = \emptyset$. D must be empty which is a contradiction since $C \subset D$ is a seperation of B. $\square$

Theorem 8.1.4. *Continuous image of a connected set is connected. i.e. If $f : X \to Y$ is continuos and X is connected, then $f(X)$ is connected.*

Proof. Any map on to its image is onto. Let's consider $f : X \to Y$ as $f : X \to f(X) = Z$ which becomes onto. (must show: Z is connected.) Suppose $Z = A \subset B$ is a seperation into two disjoint nonempty sets open in Z. Then $f^{-1}(A)$ and $f^{-1}(B)$ forms a separation for X:

- both open, since f is cont and A, B open.
- disjoint, since A, B are disjoint.
- nonempty, since f is onto and A and B are nonempty.

This contradicts to the fact that A is connected. $\square$

Theorem 8.1.5. *Finite product of connected spaces is connected.*

Proof. Let X and Y be connected spaces and $a \times b \in X \times Y$. $X \times b$ is homeomorphic to X and $x \times Y$ is homeomorphic to Y. Therefore there are also connected. The set $T_x =(X \times b) \cup (x \times Y)$ is the union of two connected sets with a common point $x \times b$. This implies T_x is also connected. But $X \times Y = \bigcup_{x \in X} T_x$ so $X \times Y$ is also a union of connected spaces and must as well be connected.
the proof for $X_1 \times \cdots \times X_n = (X_1 \times \cdots \times X_{n-1}) \times X_n$ is by induction. $\square$

Remark 8.1.6. *Product of infinitely many connected spaces may or may not be connected depending on the topology.*

Example 8.1.7. $\mathbb{R}^w = \mathbb{R}^N = \{ (x_1, x_2, \dots, x_n, \dots) | x_i \in \mathbb{R} \ i \in \mathbb{N} \}$ *can be seen as the space of sequences. With box topology on it* $\mathbb{R}^w$ *is not connected.*
$A = \{\{a_n\} \mid a_n \in \mathbb{R}$ *and* $\{a_n\}$*is a bounded sequence.*$\}$
$B = \{\{b_n\} \mid b_n \in \mathbb{R}, \{b_n\}$*is unbounded*$\}$
Clearly $A \cap B = \emptyset$. *A and B are open: Let* $a \in \mathbb{R}^w$.
$a = \{a_1, a_2, \dots\} \in U = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \dots \subset A$ *if a is bounded,*
$a = \{a_1, a_2, \dots\} \in U = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \dots \subset B$ *if a is unbounded.*
Thus A and B form a separation for $\mathbb{R}^w$ *with box topology.*

Example 8.1.8. *Consider* $\mathbb{R}^w$ *with product topology.*

Let $\tilde{\mathbb{R}}^n = \{(x_1, x_2, \dots, x_n, 0, 0, \dots) \mid x_i \in \mathbb{R}\} \subset \mathbb{R}^w$ *which is homeomorphic to* $\mathbb{R}^n$. *Assuming that* $\mathbb{R}$ *is connected, by previous theorem* $\mathbb{R}^n$ *and hence* $\tilde{\mathbb{R}}^n$ *are connected.*
Note that $0 = (0, \dots, 0, 0, \dots) \in \tilde{\mathbb{R}}^n$ *for any* $n \in \mathbb{N}$, *therefore the set* $\mathbb{R}^\infty = \bigcup_{n\in\mathbb{N}} \tilde{\mathbb{R}}^n$ *is also connected.*

Claim. $\mathbb{R}^\infty = \mathbb{R}^w$ *(and hence* $\mathbb{R}^w$ *also connected.)*

Proof. We need to show that any $a \in \mathbb{R}^w$ is a limit point of $\mathbb{R}^\infty$ i.e. any base element in the product topology containing a intersects $\mathbb{R}^\infty$.
Let $U = \prod U_i$ be a basis element containing $a = (a_1, a_2, \dots)$.
Then there is $N \in \mathbb{N}$ such that $U_i = \mathbb{R}$ for $i > N$ (see Theorem 7.1.4)
The point $x = (a_1, a_2, \dots, a_n, 0, \dots) \in \mathbb{R}^\infty \cap U$ since
$a_i \in U_i$ for all i and $0 \in U_i$ for $i > N$ □

Question. *If* $\tau_1 \subset \tau_2$ *are two topologies on X are the following true or false?*

i. (X, τ_1) *connected* $\implies$ (X, τ_2) *connected.*

ii. (X, τ_2) *connected* $\implies$ (X, τ_1) *connected. environment.*

8.2 Connected Subspaces of $\mathbb{R}$

A set X is simply ordered if there is a relation $\leq$ such that:

i. $a \leq b$ and $b \leq a$ imply $a = b$ (antisymmetry)

ii. $a \leq b$ and $b \leq c$ imply $a \leq c$ (transitivity)

iii. $a \leq b$ or $b \leq a$ (comparability)

(One can also add reflexivity $a \leq a$)

Definition 8.2.1. *A simply ordered set L having more than one element is a linear continuum if*

i. L has the least upper bound property

ii. If $x < y$ there exists $z \in L$ s.t. $x < z < y$.

Recall that a set has the l.u.b property if every nonempty subset with an upper bound has a least upper bound. (supremum)

Theorem 8.2.2. *If L is a linear continuum in the order topology then L is connected and so are intervals and rays in L.*

Proof. see Munkres. $\square$

Corollary 8.2.3. $\mathbb{R}$ *is connected and so are intervals and rays in* $\mathbb{R}$

Theorem 8.2.4. *(Intermediate Value Theorem) Let X be a connected space and Y have order topology. If $f : X \longrightarrow Y$ is continuous, $a, b \in X$ and r is between $f(a)$ and $f(b)$, then there is $c \in X$ such that $f(c) = r$.*

Proof. The sets $A = f(X) \cap (-\infty, r)$ and $B = f(X) \cap (r, \infty)$ are disjoint and nonempty because one contains $f(a)$ and the other contains $f(b)$. A and B are open in $f(X)$ because they are intersections of open rays with $f(X)$. If there is no $c \in X$ with $f(c) = r$, then $f(X) = A \cup B$ is a separation for $f(X)$. This contradicts to the fact that continuous image of a connected set is connected. $\square$

8.2.1 Path Connected Spaces

Definition 8.2.5. *Given points x and y of X, a path in X from x to y is a continuous function $f : [a, b] \to X$ of some closed interval of $\mathbb{R}$ into X such that $f(a) = x$ and $f(b) = y$. A space X is path-connected if every pair of points in X can be joined by a path in X.*

Observe : *Path connected $\Rightarrow$ connected.* Suppose not and let $A \cup B$ be a separation for the path connected space X. Let $f : [a, b] \to X$ be any path in X. Since f is continuous and $[a, b]$ is connected, $f([a, b])$ is also connected. So it must either lie entirely in A or entirely in B. This means we cannot find a path from a point of A into a point of B which is a contradiction.

The converse is not true. The "topologist's sine curve" is connected but not path-connected.

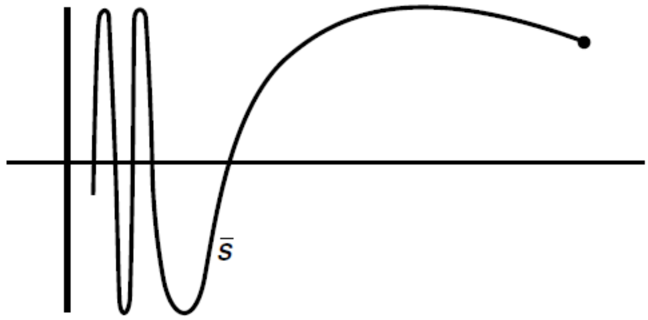


Figure 8.1: Topologist's sine curve

Example 8.2.6. *Let* $S = \{(x, y) \cap \mathbb{R}^2 \mid 0 < x \leq 1\ ,\ y = \sin(\frac{1}{x})\}$

Since $f(x) = \sin(\frac{1}{x})$ *is continuous on* $(0, 1]$*,* S *is connected. Therefore* $\overline{S}$ *(called the topologist's sine curve) is also connected.* $\overline{S}$ $=$ $S\cup$ $\{$ *(0,y)* $\mid y \in [-1, 1]\}$*.* *<u>Claim</u>* : $\overline{S}$ *is not path connected.*

Proof. Suppose not and let $f : [a, c] \to \overline{S}$ be a path from $(0, 0)$ to a point of S. Since f is continuous and $\{0\}$ x $[-1, 1]$ is closed, $f^{-1}(\{0\}$ x $[-1, 1])$ is closed and hence has a largest element b. Then $f : [b, c] \to \overline{S}$ is a path that maps b into $\{0\}$ x $[-1, 1]$ and other points of $[b, c]$ to points of S. For convenience replace $[b, c]$ by $[0, 1]$ and let $f(t) = (x(t), y(t))$. Then $x(0) = 0$ and when $t > 0$ we have $x(t) > 0$ and $y(t) = \sin\frac{1}{x(t)}$. Construct a sequence $\{t_n\}$ as follows: For $n \in \mathbb{N}$ let u be $0 < u < x(\frac{1}{n})$ s.t. $\sin(\frac{1}{u})$ $=(-1)^n$. By IVT, since $x(t)$ is continuous, there's t_n with $0 < t_n < \frac{1}{n}$ s.t. $x(t_n)$ $=u$. Thus the sequence $\{t_n\}$ is a sequence of points converging to 0. But $y(t)$= $\sin(\frac{1}{x(t_n)})$ $= \sin(\frac{1}{u}) = (-1)^n$ which is divergent. This contradicts to the fact that f is continuous. □

8.2.2 Connected Components

Definition 8.2.7. *Given* X*, define an equivalance relation on* X *by setting* $x \sim y$ *if there is a connected subspace of* X *containing both* x *and* y*. The equivalence classes are called the components of* X*.*

Recall that the union of two connected subsets, having a point in common, must also be connected.

Theorem 8.2.8. *The components of* X *are connected disjoint subspaces of* X *whose union is* X*, such that each nonempty connected subspace of* X *intersects only one of them.*

Definition 8.2.9. *Define an equivalence relation $\sim$ on X by $x \sim y$ if and only if there is a path in X from x to y. The equivalence classes are called path components.*

A path from x to y was a continuous function $f : [a, b] \to X$ such that $f(a) = x$ and $f(b) = y$. For convenience let's take $[a, b] = [0, 1]$ (any two closed interval in $\mathbb{R}$ are homeomorphic). Let's see that $\sim$ is an equivalence relation:

i) (Reflexive) $f : [0, 1] \to \{x\}$ defined as $f(t) = x$ the constant map is a path from x to x i.e. $x \sim x$

ii) (Symmetric) If $f : [0, 1] \to X$ is a path from x to y, then $f(1 - t)$ is a path from y to $x \Rightarrow$ "$x \sim y \Rightarrow y \sim x$"

iii) (Transitive) Let $f : [0, 1] \to X$ and $g : [0, 1] \to X$ be paths from x to y and from y to z, respectively.
Define
$$h(t) = \begin{cases} f(2t) & 0 \leq t \leq \frac{1}{2} \\ g(2t - 1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$
which is continuous by pasting lemma (f and g coincide on the intersection). Hence h is a path from x to $z \Rightarrow x \sim z$

Example 8.2.10. *The topologist's sine curve has one component but two path components.*

Example 8.2.11. $\mathbb{Q}$ *has components consisting of single points.*

Chapter 9

Compact Spaces

Compactness is not as natural as connectedness but it is an important property of some Euclidian subspaces. The proofs of many important results in analysis require compactness.

Definition 9.0.1. *A collection $\mathcal{A}$ of subsets of a space X is a cover to X (or covering), if the union of elements of $\mathcal{A}$ is equal to X.*
i.e. $\mathcal{A} = \{A_\alpha \mid \alpha \in \Lambda, A_\alpha \subset X\}$ *and* $X = \bigcup_{\alpha \in \Lambda} A_\alpha$

If elements of $\mathcal{A}$ are open in X, then $\mathcal{A}$ is an open covering of X.

Definition 9.0.2. *If a subcollection $\mathcal{B}$ of a covering $\mathcal{A}$ is a cover itself, then $\mathcal{B}$ is said to be a subcover of $\mathcal{A}$.*

Definition 9.0.3. *A space X is said to be compact if every open covering has a finite subcover.*

Remark 9.0.4. *Negating the above definition we get: "X is not compact if it has an open cover without a finite subcover".*

Example 9.0.5. $\mathbb{R}$ *is not compact.*

Consider $\mathcal{A} = \{(n, n+2) \mid n \in \mathbb{Z}\}$ *which is an open cover for* $\mathbb{R}$. *But no finite subcollection of* $\mathcal{A}$ *can cover* $\mathbb{R}$.

Example 9.0.6. *If X is a finite set, then any open cover will be finite. Hence its subcovers are also finite. So X must be compact.*

Example 9.0.7. $X = \{0\} \in \{\frac{1}{n} \mid n \in \mathbb{N}\}$ *is compact.*

Proof. Suppose $\mathcal{A}$ is an open cover for X and let $U \in \mathcal{A}$ be any open element of $\mathcal{A}$ containing 0.

All but finitely many points of X are in U. If $U_1, U_2, ..., U_k$ are elements of $\mathcal{A}$ containing these points that are not in U, then $\mathcal{B} = \{U, U_1, U_2, U_3, ..., U_k\}$ is a finite subcover. $\square$

Example 9.0.8. *If $K = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ then K is not compact.*

Proof. For each $n \in \mathbb{N}$ let $\epsilon_n = \frac{1}{2}(\frac{1}{n} - \frac{1}{n+1}) = \frac{1}{2n(n+1)}$ then $U_n = (\frac{1}{n} - \epsilon_n, \frac{1}{n} + \epsilon_n) \cap K = \{\frac{1}{n}\}$ is an open set in the subspace topology of K that contains only $\frac{1}{n}$. The open cover $\mathcal{A} = \{U_1, U_2, ...\}$ has no subcover. $\square$

Example 9.0.9. *The interval $(0,1]$ is not compact. $\mathcal{A} = \{(\frac{1}{n}, 1] \mid n \in \mathbb{N}\}$ is an open cover without a finite subcover.*

Example 9.0.10. $(0,1)$ *is not compact.*

Example 9.0.11. $[0,1]$ *is compact.*

Lemma 9.0.12. *A subspace Y of X is compact iff every covering of Y by sets open in X has a finite subcovering of Y.*

Proof. ($\Rightarrow$) Let Y be compact (i.e. every open (in Y) cover of Y has a finite subcover) and $\mathcal{A} = \{A_\alpha \mid \alpha \in j\}$ be a covering of Y by sets open in X.
Then $\{A_\alpha \cap Y \mid \alpha \in j\}$ is an open cover of Y by sets open in Y.
Since Y is compact, this open cover must have a finite subcover:
$\{A_{\alpha_1} \cap Y, A_{\alpha_2} \cap Y, ..., A_{\alpha_n} \cap Y\}$.
Thus, $\{A_{\alpha_1}, A_{\alpha_2}, ..., A_{\alpha_n}\}$ is a subcover of $\mathcal{A}$ for Y

($\Leftarrow$) Let $\mathcal{A} = \{A_\alpha\}$ be a covering of Y by sets open in Y. For each α, choose a set B_α open in X s.t. $A_\alpha = B_\alpha \cap Y$.
The collection $\mathcal{B} = \{B_\alpha\}$ is a covering of Y by sets open in X. By hypothesis some fine subcollection $\{B_{\alpha_1}, ..., B_{\alpha_n}\}$ covers Y. Then $\{A_{\alpha_1}, ..., A_{\alpha_1}\}$ is a subcover of $\mathcal{A}$ for Y. $\square$

Theorem 9.0.13. *Every closed subspace of a compact space is compact.*

Proof. Let Y be a closed subspace of the compact X. Given a covering A of Y by sets open in x (By above lemma we need to find a finite subcollection of A covering Y) the union $\mathcal{B} = A \cap X - Y$ is an open cover for the compact space X. If this subcollection contains X-Y discard X-Y, otherwise leave it as it is. $\square$

Theorem 9.0.14. *Continuous image of a compact space is compact.*

Proof. Let f:X→ Y be continuous and X be compact. Let $\mathcal{A}$ be a covering of the set $f(X)$ by sets open in Y. Then $\{f^{-1}(A) \mid A \in \mathcal{A}\}$ is a collection of sets covering X. These are open in X because f is continuous. Hence finitely many of them , say $f^{-1}(A_1)$, . . . ,$f^{-1}(A_n)$ cover X. Then A_1, . . . ,A_n cover $f(X)$. □

Theorem 9.0.15. *Let X be a simply ordered set having the least upper bound property. In the order topology, each closesd interval in X is compact .*

Theorem 9.0.16. *Every closed interval in $\mathbb{R}$ is compact.*

Theorem 9.0.17. ***(Heine-Borel Theorem)****A subspace of $\mathbb{R}^n$ is compact if and only if it is closed and bounded in the euclidean metric or the square metric.*

Example 9.0.18. $K = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ *is not compact because it is not closed.* 0 *is its limit point but* $0 \notin K$*.*

Example 9.0.19. S^{n-1} *and* $D^n = B^n$ *in* $\mathbb{R}^n$ *are compact because they are closed and bounded.*

Example 9.0.20. $\mathbb{R}^n$ *is not bounded hence* $\mathbb{R}^n$ *is not compact.*

Example 9.0.21. $\mathbb{R}^n$=S^{n-1}-$\{point\}$ *is not compact.*

Example 9.0.22. $\{(x, \frac{1}{x}) \mid 0 < x \leq 1\} \subset \mathbb{R}^2$ *is closed but not bounded. Therefore it cannot be compact.*

Example 9.0.23. $\{(x, y) \mid y = \sin\frac{1}{x},\ x \in (0, 1]\}$ *is bounded but not closed* $\Rightarrow$ *not compact.*

Theorem 9.0.24. ***(Extreme Value Theorem)*** *Let $f : X \to Y$ be continuous where Y is an ordered set in the ordered topology. If X is compact, then there exists points c and d in X such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.*

In Calculus courses we have seen this theorem with X as a closed interval and $Y = \mathbb{R}$.

Proof. Since f is continuous and X is compact, the set $A = f(X)$ is compact. We must show that A has a largest element M and a smallest element m.

Suppose A has no largest element. Then the collection $\{(-\infty, a) \mid a \in A\}$ forms an open covering of A . Since A is compact this cover must have a finite subcover:
$\mathcal{B} = \{(-\infty, a_1), (-\infty, a_2), ...(-\infty, a_n)\}$. If a_i is the largest of the elements $a_1, ..., a_n$, then a_i does not belong to these sets. This contradicts to the fact that $\mathcal{B}$ is a cover for A. With a similar argument we can show the existence of a smallest element of A. □

Theorem 9.0.25. ***(Tychonoff theorem))*** *An arbitrary product of compact spaces is compact in the product topology.*

Thus, a product space is compact if and only if all factor spaces are compact. Note that the theorem is true even for uncountable product of spaces.

Example 9.0.26.

i. The n-dimensional torus $T^n = (S^1)^n = S^1 \times \cdots \times S^1$ is compact since S^1 is compact.

ii. $[0,1] \times [0,1)$ is not compact since $[0,1)$ is not closed and hence not compact.

iii. $[0,1]^\omega$ is compact.

Chapter 10

Separation Axioms

Consider the sequence $\{a_n \mid a_n = a \ for \ all \ n \in \mathbb{N}\} = \{a, a, a, ...\}$ of the space $X = \{a, b\}$. If X is endowed with the trivial topology $\tau = \{\phi, X\}$, then this sequence converges to a, and also to b ($\lim a_n = x \in X$ iff for every neighborhood U of x there's $N \in \mathbb{N}$ such that $a_n \in U$ for $n \geq N$).

In Calculus courses, we used to have convergent sequences with unique limits or ,if existed, we expect $\lim\limits_{x \to x_0} f(x)$ to be a unique number.The above example suggests that the trivial topology does not have enough number of open sets to separate two points, even on a small set like $X = \{a, b\}$.

Definition 10.0.1. *A topological space X is a T_0-space if whenever x and y are distinct points in X, then there is an open set containing one and not the other.*

Example 10.0.2. *Trivial topology ($\tau = \{\phi, X\}$) on any set with more than one point is not T_0.*

Example 10.0.3. *Subspaces and products of T_0 spaces are T_0*

Definition 10.0.4. *A topological space X is a T_1-space if, whenever x and y are distinct points in X, then there is a neighborhood of each not containing the other.*

Clearly, every T_1-space is T_0.

Example 10.0.5. $\tau = \{\phi, \{a\}, \{a, b\}\}$ *is T_0 but not T_1 on $X = \{a, b\}$*

Theorem 10.0.6. *The following are equivalent, for a topological space X:*

a) X is T_1,

b) One-point sets (singletons) are closed in X, *c) Each subset of X is the intersection of the open sets containing it.*

Proof. (a⇒b): If X is T_1 and $x \in X$, then each $y \neq x$ has a neighborhood U_y disjoint from $\{x\}$. Then $X - \{x\} = \bigcup_{y \neq x} U_y$ is open and hence $\{x\}$ is closed.

(b⇒c): Let $A \subset X$ be a subset. For each $x \in X - A$ the set X - {x} is open by the hypothesis, and $A \subset X - \{x\}$. Then: $A = X - \bigcup_{x \notin A} \{x\} = \bigcap_{x \in X - A} (X - \{x\})$ (De Morgan).

(c⇒a): By c), the set $\{x\}$ is also intersection of open sets containing x: i.e $\{x\} = \bigcap U_\alpha$ where $x \in U_\alpha$ for all α. Thus, if $y \neq x$ then for some α' we must have $y \notin U_{\alpha'}$. □

The real importance of T_1-spaces lies in the observation above: one-point sets are closed in T_1-spaces.

Theorem 10.0.7. *Let A be a subset of T_1-space X. Then $x \in X$ is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.*

Proof. (⇐): Definition of limit requires that every neighborhood of x should intersect A in some point other than x itself. Since neighborhoods of x intersect A in infinitely many points, it must intersect in some point other than x.

(⇒): Suppose U is a neighborhood of x that intersects A in finitely many points and let $\{x_1, ..., x_m\} = U \cap (A - \{x\})$. This set is closed since X is T_1. Therefore $X - \{x_1, ..., x_m\}$ is open. Setting $V = U \cap (X - x_1, ..., x_m)$ we obtain a neighborhood of x which does not intersect $A - \{x\}$. This contradicts to the fact that x is a limit point of A. □

10.1 Hausdorff Spaces

A space X is T_2 (or Hausdorff) if, whenever x and y are distinct points of X, there are disjoint open sets U and V in X with $x \in U$ and $y \in V$.

In order to have unique limits for convergent sequences in X, the topology on X must be Hausdorff.

Example 10.1.1. *Let X be an infinite set with the cofinite topology. (i.e. a subset $A \subset X$ is open if and only if X-A is finite) Then, the closed sets are X and finite sets, in particular one-point sets are closed. Therefore X is a T_1-space. Non-empty open sets cannot be disjoint, hence X is not T_2.*

Example 10.1.2. *Every metric space is Hausdorff. If x and y are distinct, then let $\varepsilon = d(x, y) > 0$ be distance between them. The ε-balls $B_d(x, \frac{\varepsilon}{2}) = U$ and $B_d(y, \frac{\varepsilon}{2}) = V$ are open disjoint sets containing x and y, respectively.*

Theorem 10.1.3. *If X is a Hausdorff space, then a sequence of points of X converges to at most one point in X.*

Proof. Let $\{x_n\}$ be a sequence converging to $x \in X$. If $y \neq x$, let U and V be disjoint neighborhoods of x and y respectively. Since U contains all but finitely many terms of $\{x_n\}$, the set V cannot. Therefore $\{x_n\}$ cannot converge to y. $\square$

Theorem 10.1.4. *Every simply ordered set is a Hausdorff space in order topology. Product of Hausdorff spaces is a Hausdorff space. Subspace of a Hausdorff space is Hausdorff.*

10.1.1 Properties of Hausdorff Spaces

Theorem 10.1.5. *Every compact subspace of Hausdorff space is closed.*

Proof. Let Y be a compact subspace of the T_2-space X. Let $x_0 \in X - Y$. We need to show that x_0 has a neighborhood $U \subset X - Y$ (or $U \cap Y = \emptyset$) so that $X - Y$ is open. Since X is Hausdorff, for each $y \in Y$ there are disjoint neighborhoods U_y and V_y of x_0 and y respectively. Then $\mathcal{A} = \{V_y | y \in Y\}$ is an open cover of Y. Since Y is compact $\mathcal{A}$ must have a finite subcover $\{V_{y_1}, ..., V_{y_n}\}$. Setting $V = V_{y_1} \cup ... \cup V_{y_n}$ and $U = U_{y_1} \cup ... \cup U_{y_n}$ we obtain disjoint open sets U and V where $Y \subset V$ and $x_0 \in U$. $\square$

Theorem 10.1.6. *Continuous image of a compact space is compact.*

Proof. Let $f : X \to Y$ be continuous, X be compact and $\mathcal{A}$ be an open cover for $f(X)$. Then the collection $\{f^{-1}(A_\alpha) | A_\alpha \in \mathcal{A}\}$ is an open cover for X, since f is continuous. Hence finitely many of them $\{f^{-1}(A_1)\}$,...,$\{f^{-1}(A_n)\}$ cover X. This implies $A_1, ..., A_n$ cover $f(X)$. $\square$

Theorem 10.1.7. *Let $f : X \to Y$ be $1-1$, onto and continuous. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

Proof. We need to show that f^{-1} is also continuous. This is equivalent to f being a closed function i.e. $f(A)$ must be closed for every closed subset $A \subset X$. Closed subsets of compact spaces are compact. Hence A is compact. Since f is continuous, $f(A)$ is also compact. By the above theorem, $f(A)$ must be closed. $\square$

Chapter 11

Countability Properties

A space X is said to have a countable basis at x, if there is a countable collection $\mathcal{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathcal{B}$. A space that has a countable basis at each of its points is said to be first-countable.

Example 11.0.1. *Every metrizable space is first-countable. Let x be a point of the metric space (X, d) and set $\mathcal{B} = \{B_d(x, \frac{1}{n}| n \in \mathbb{N}\}$ which is a countable collection of concentric ε-balls. If $U \subset X$ is any neighborhood of x then there is $\varepsilon > 0$ s.t. $B_d(x, \varepsilon) \subset U$. For any natural number $N \in \mathbb{N}$ s.t. $N > \frac{1}{\varepsilon}$ we have $B_d(x, \frac{1}{N}) \subset B_d(x, \varepsilon) \subset U$.*

Definition 11.0.2. *The collection $\mathcal{B}$ in the above definition is also called a local basis at x. Therefore a first-countable space must have a local basis at every point in it.*

Theorem 11.0.3. *Let X be a topological space;*

(a) If there is a sequence of points of $A \subset X$ converging to $x \in X$, then $x \in \overline{A}$; the converse holds if X is first-countable.

(b) Let $f : X \to Y$ be continuous and $\{x_n\}$ converges to x. Then $\{f(x_n)\}$ converges to $f(x)$. The converse holds if X is first-countable.

Definition 11.0.4. *If a space X has a countable basis for its topology, then X is said to be second-countable. Clearly: second countable $\Rightarrow$ first countable. (Countable basis $\mathcal{B}$ for X can be taken as a local basis at any $x \in X$)*

Example 11.0.5. *$\mathcal{B} = \{(a, b)| a, b \in \mathbb{Q}\}$ is a countable basis for the real line $\mathbb{R}$. Thus $\mathbb{R}$ is second-countable.*

Theorem 11.0.6. *Subspaces and countable products of first-countable (respectively second-countable) spaces are also first-countable (respectively second-countable).*

Definition 11.0.7. *A subset A of a space X is said to be dense in X if $\overline{A} = X$.*

Example 11.0.8. $\mathbb{Q}$ *is dense in* $\mathbb{R}$ *i.e.* $\overline{\mathbb{Q}} = \mathbb{R}$.

Theorem 11.0.9. *Let X be a second-countable space.*

(a) Every open cover of X has a countable subcover.

(b) There is a countable dense subset of X.

Proof. Let $\{B_n\}$ be a countable basis for X.

(a) Let $\mathcal{A}$ be an open cover of X. For $n \in \mathbb{N}$, if possible, choose $A_n \in \mathcal{A}$ s.t. $B_n \subset A_n$. The collection $\mathcal{A}' = \{A_n | n \in j \subset \mathbb{N}\}$ is countable and is a subcover of $\mathcal{A}$: Let $x \in X$ and choose $A \in \mathcal{A}$ containing x. Since A is open, there is B_n s.t. $x \in B_n \subset A$. This means $n \in j$ because it is possible for B_n to find an element of $\mathcal{A}$ (A in this case) containing B_n. Thus A_n is defined and hence $x \in B_n \subset A_n \Rightarrow \mathcal{A}'$ covers X.

(b) From every nonempty basis element B_n choose a point x_n. Let D be the set of such points. Then D is dense in X. If $x \in X$ then every basis element containing x intersects $D \Rightarrow x \in \overline{D}$.

□

Definition 11.0.10. *A space for which every open cover has a countable subcover is called Lindelöf space.*

Definition 11.0.11. *A space having a countable dense subset is said to be separable.*

The above theorem says that a 2^{nd}countable space is both Lindelöf and separable.

Example 11.0.12. $\mathbb{R}_l$ *is* 1^{st}*countable, Lindelöf and separable but not* 2^{nd} *countable:*

Recall that sets of the form $[a, b)$ constitute a basis for $\mathbb{R}_l$.

1st countable : Given $x \in \mathbb{R}_l$, $\beta_x = \{[x, x+(1/n)] \mid n \in \mathbb{N}\}$ is a countable basis at x.

2nd countable : For any basis β for $\mathbb{R}_l$, choose an element of B such that $x \in \beta_x \subset [x, x+1)$ for every $x \in \mathbb{R}_l$. If $x \neq y$, then $\beta_x \neq \beta_y$ because $x = \inf \beta_x$ and $y = inf \beta_y$. Therefore β must be uncountable.

Lindelöf : See Munkres.

Remark 11.0.13. *Product of Lindelöf Spaces need not be Lindelöf.*

Remark 11.0.14. *Subspaces of Lindelöf Spaces need not be Lindelöf.*

11.1 Uncountability

Recall that an infinite set X is countable if there is a surjection:
$f : \mathbb{N} \to X$ or there is an injection $g : X \to \mathbb{N}$
The result below uses topological properties of $\mathbb{R}$ to show no such f exists.

Definition 11.1.1. *If X is a space, a point $x \in X$ is an isolated point of X, if the one-point set $\{x\}$ is open in X.*

Theorem 11.1.2. *Let X be a nonempty compact Hausdorff space. If X has no isolated points, then X is uncountable.*

Corollary 11.1.3. *Every closed interval in $\mathbb{R}$ is uncountable.*

Chapter 12

Regular and Normal Spaces

Let X be a topological space.

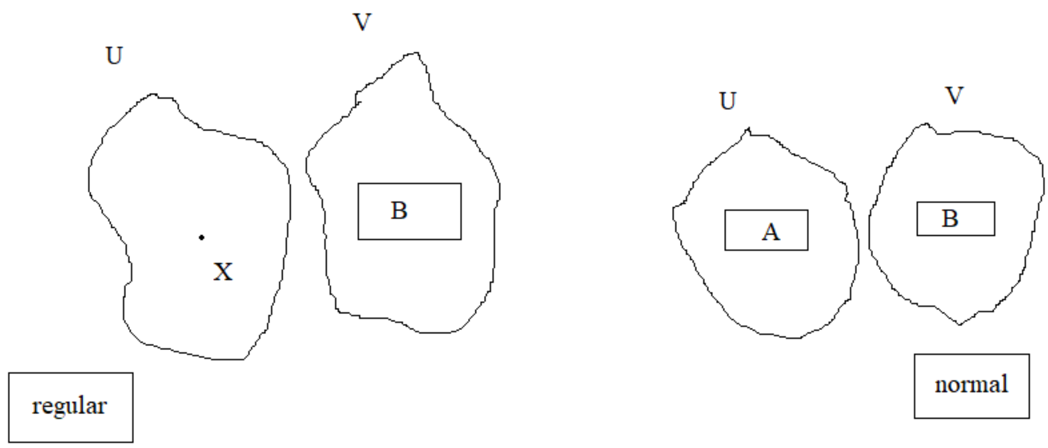
Definition 12.0.1. *X is a regular space if, for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively.*

Definition 12.0.2. *X is a normal space if, for each pair A, B of disjoint closed sets of X, there are disjoint open sets containing A and B.*

Remark 12.0.3. *If X is a T_1-space so that one-point sets are closed, then*

i. it is a T_3 space if it is also regular, and

ii. it is a T_4 space if it is also normal.



$T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$

Theorem 12.0.4. *a) X is regular if and only if for any $x \in X$ and a neighborhood U of x there is a neighborhood V of x such that $\overline{V} \subset U$*

b) X is normal if and only if for any closed set A and an open set U containing A there is an open set V such that $A \subset V \subset \overline{V} \subset U$

Proof. (a) ($\Rightarrow$) Let X be regular and suppose that the point x and the neighborhood U of x are given. Let $B = X - U$ which is a closed set not containing x. Then there are disjoint open sets V and W containing x and B respectively. □

claim: $\overline{V}$ and B are disjoint.

Proof. A point $y \in B$ can not be a limit point of V, since W is a neighborhood of y disjoint from V. Therefore, $\overline{V} \subset U$.

($\Leftarrow$) Let $x \in X$ and B be a closed set not containing x. Set $U = X - B$ which is a neighborhood of x. Then there is a neighborhood V of x such that $\overline{V} \subset U$. The open sets V and X - $\overline{V}$ are disjoint open sets containing x and B respectively. Thus X is regular.

□

Theorem 12.0.5. *Subspaces and products of Hausdorff spaces are Hausdorff.*

Theorem 12.0.6. *Subspaces and products of regular spaces are regular.*

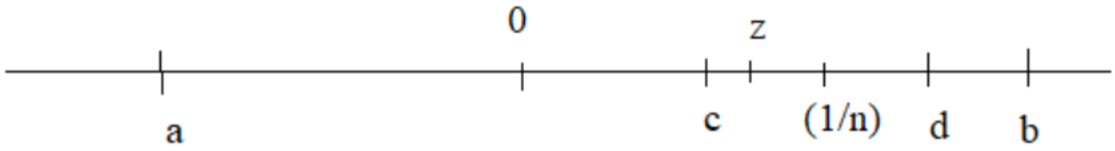
Remark 12.0.7. *The above theorem does not hold for normal spaces.*

Example 12.0.8. *The space $\mathbb{R}_K$ is Hausdorff but not regular. For $K = \{(1/n) | n \in \mathbb{N}\}$ the sets of the form $(a, b) - K$ with open intervals (a, b) form a basis for $\mathbb{R}_K$. Since $\mathbb{R}_K$ is finer than $\mathbb{R}_{st}$ obviously $\mathbb{R}_K$ is T_2.*

Claim : $\mathbb{R}_K$ is not regular.

Proof. The set K is closed and $0 \notin K$. We want to show that we cannot separate 0 and K. Suppose the open sets U and V contain 0 and K respectively. Basis elements containing 0 and lying in U must be of the form $(a, b) - K$, otherwise they intersect K. For large enough n, let $(1/n) \in (a, b)$. Since V is a neighborhood of $(1/n)$, a basis element containing $(1/n)$ and lying in V must be of the form (c, d). Choose z such that $z > \max\{c, (1/(n + 1))\}$. Then z belongs to both U and V, so they are not disjoint.

□



Example 12.0.9. $\mathbb{R}_l$ *is normal.*

Let A and B be disjoint closed sets in $\mathbb{R}_l$. For each point $a \in A$ choose a basis element $[a, x_a)$ not intersecting B, and for each $b \in B$ choose $[b, x_b)$ not intersecting A. Then the open sets
$U = \bigcup_{a \in A} [a, x_a)$ *and* $U = \bigcup_{b \in B} [b, x_b)$
are disjoint open sets containing A and B, respectively.

Example 12.0.10. $\mathbb{R}_l^2$ *is not normal.*

12.1 Normal Spaces

Theorem 12.1.1. *Every regular space with a countable basis is normal.*

Theorem 12.1.2. *Every metrizable space is normal*

Proof. Let (X, d) be a metric spaces. Let A and B be disjoint closed subsets of X. For each $a \in A$ choose ϵ_a so that $B_d\left(a, \epsilon_a\right) \cap B = \emptyset$. Similarly for each $b \in B$, choose ϵ_b so that $B_d\left(b, \epsilon_b\right) \cap A = \emptyset$

$U = \bigcup_{a \in A} B\left(a, \frac{\epsilon_a}{2}\right)$ and $U = \bigcup_{b \in B} B\left(b, \frac{\epsilon_b}{2}\right)$

Then U and V are open sets containing A and B, respectively. They are also disjoint. If $z \in U \cap V$, then $z \in B_d(a, \frac{\epsilon_a}{2}) \cap B_d(b, \frac{\epsilon_b}{2})$ for some $a \in A$ and $b \in B$. B_y the triangle inequality $d\left(a, b\right) < \frac{\epsilon_a + \epsilon_b}{2}$. If $\epsilon_a \leq \epsilon_b$, then $d\left(a, b\right) < \epsilon_b$ which means $a \in B_d(b, \epsilon_b)$ If $\epsilon_b \leq \epsilon_a$, then $d\left(a, b\right) < \epsilon_a$ implying that $b \in B_d(a, \epsilon_a)$ Both cases are impossible $\square$

Theorem 12.1.3. *Every compact Hausdorff space is normal*

Proof. Let X be a compact Hausdorff space. $\square$

<u>Claim:</u> X is regular.

<u>Proof:</u> Let $B \subset X$ be a closed set and $x \in X$ with $x \notin B$. Then B is compact. Since X is Hausdorff, fore every $b \in B$ choose open sets $b \in U_b$ and $a \in V_b$

s.t. $U_b \cap V_b = \emptyset$. Then $\{U_b | b \in B\}$ is an open cover for B, which should have a finite subcover, say $\{U_{b_1}, U_{b_2} ... U_{b_n}\}$. If we let

$U = \bigcup_{i=1}^{n} U_{b_i}$ and $V = \bigcap_{i=1}^{n} V_{b_i}$,

then we find disjoint open sets containing B and x respectively. ■

Claim: X is normal.

Proof: The proof is essentially the same as above. Let A and B disjoint closed sets of X. For each point $a \in A$, choose disjoint open sets U_a and V_b containing a and B, respectively (We can do this, since X is regular). Since A is compact its open cover $\{U_a\}$ has a finite subcover $\{U_{a_1}, U_{a_2} ... U_{a_n}\}$. Then,

$U = \bigcup_{i=1}^{n} U_{a_i}$ and $V = \bigcap_{i=1}^{n} V_{a_i}$

are disjoint open sets containing A and B respectively ■

Theorem 12.1.4. *Every well-ordered set in the order topology is normal (In fact, every order topology is normal)*

12.1.1 Urysohn's Lemma

This is a deep result used in proving a number of important theorems. Its proof involves an original idea which is beyond the scope of this course (See Munkres).

Theorem 12.1.5. *(Urysohn's Lemma) Let X be a normal spaces. Let A and B be disjoint closed subsets of X. Let $[a, b]$ be a closed interval in the real line. Then there exist a continuous map $f : X \to [a, b]$ such that $f(A) = a$ and $f(B) = b$.*

As a consequence we have the Urysohn metrization theorem.

Theorem 12.1.6. *Every regular space X with a countable basis is metrizable.*

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