Large Deflections of Bi-directional Functionally Graded Cantilever Beams

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***Abstract* - The large deflections of a bi-directional cantilever beam that obeys modified Ludwick's constitutive law and made of FGM is analyzed as a layered structure in this study. The nonlinearly elastic cantilever beam is subjected to a concentrated force at the free end. The modified Ludwick type of material model and functionally graded material properties are defined by using Marlow's material model in finite element analysis. The results show that the number of layers is one of the most important properties in the evaluation of both normal stress distributions along the thickness direction and deflections of the free ends of slender beams. The effect of the number of layers on the stress distributions on the beam is obvious when compared particularly the single-layered FGM with any of the multi-layered FGMs.**

***Keywords* - Large deflections, functionally graded materials, cantilever beams, Ludwick's law, nonlinearly elastic.**

I. Introduction

FUNCTIONALLY graded materials (FGMs) have long been a material of great interest in a wide range of fields. Recent trends in 3-D printing technologies have led to a proliferation of studies that examine FGMs. In addition, recent years have witnessed a growing academic interest in both FGMs and non-linearly elastic materials. Thus, FGMs have been an object of research that examine the behavior of beams subjected to various loads [1]. During the last decade, the combination of FGMs and nonlinearly elastic materials has been at the center of much attention. Many researchers have considered not only geometrical nonlinearity but also material nonlinearity. Much of the literature since the mid-1990s emphasizes the analytical solutions of nonlinear beams subjected to large deflections [2-4].

Lewis and Monasa [5,6] examined a thin cantilever beam under a transverse end load and a moment in terms of large deflections of the free end. Baykara et al. [7] considered nonlinear bi-modulus material behavior to investigate a thin cantilever beam with an end moment. A large buckling analysis in fibrous materials using Ludwick type material model was examined by Jung and Kang [8]. Kang and Li [9,10] took into account large and small deflections of nonlinearly elastic functionally graded beams (FGBs). The large deflections of a slender, non-homogeneous beam under combined loads were studied by Sitar et al. [11]. In that study, the beam was modelled as a functionally graded material and discretized with a certain number of nonlinearly elastic layers that obey generalized Ludwick's constitutive law.

Hacıoğlu [12] examined the variation of Young's modulus relating to power-law and geometrical nonlinearity on the deflection. The results of that study showed that gradient indexes and material constant in Ludwick's law significantly affect the bending strength of an FGM beam. Hacıoğlu and Baykara [13] examined the nonlinear functionally graded beam subjected to concentrated and combined loads. Moreover, Hacıoğlu et al. [14] studied large deflections of a cantilever beam which is made of nonlinearly elastic, modified Ludwick's type of material using FEA. Marlow's material model was used in Ref. [14] to combine direction dependent material properties in the functionally graded material and nonlinearity from modified Ludwick's law.

However, such studies remain narrow in focus dealing only with one-directional functionally graded beams. In all the studies reviewed here, the modelling of the FGM beam which is layered in the thickness direction is recognized as significant. Karamanlı [15,16] investigated the elastostatic behavior of two-directional functionally graded beams under different sets of boundary conditions. A bending analysis study was presented to examine the effects of different sets of boundary conditions on FGM beams [17].

In this study, the large deflections of a nonlinearly elastic bi-directionally functionally graded cantilever beam subjected to end load are presented. A combination of the bi-directional FGM model and nonlinearity of material properties based on modified Ludwick's law is introduced. The material properties vary through both thickness and length directions with respect to the function of FGMs. Firstly, we present the effect of the number of layers in thickness direction on the deflections at the end of the beam. Then, bi-directional layer is applied and the effect of both thickness and length directions are examined.

II. Problem Definition and Modeling

A slender cantilever beam of length, L through x axis and the rectangular cross-section of thickness, h through y axis and width, b is used as seen in Fig. 1. The bi-directional functionally graded beam is composed of several rigidly bonded layers which are named as nh and nl in thickness (h) and length (L) directions, respectively. Layers are assumed to be incompressible. F_y is the point load applied vertically at the free end of the beam as defined in Fig. 1.

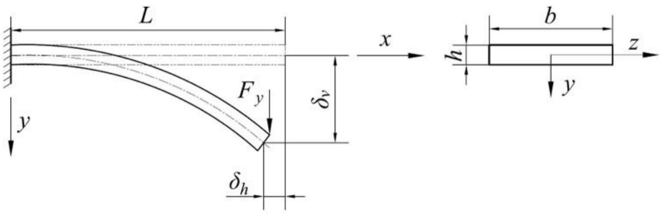

Figure 1: Geometric details of the beam.

The modified Ludwick's constitutive model that governs the stress-strain relationship is given by the following expression [18],

$$\sigma = \text{sign}(\varepsilon)E\left((|\varepsilon| + \varepsilon_0)^{\frac{1}{k}} - \varepsilon_0^{\frac{1}{k}}\right) \qquad \text{for all } \varepsilon, \tag{1}$$

where, σ and ε defines normal stress and strain respectively. E, ε_0, and k are material constants and sign(ε) represents the symbol function which has a value of 1 or -1 in a case of tension or compression. Here, Ludwick's constitutive relation is obtained by setting $\varepsilon_0 = 0$ while Hooke's constitutive law is obtained by setting $k = 1$. In order to define the material model of modified Ludwick type material, Marlow's first-invariant constitutive model was adapted in this study [19]. In Marlow's form, it is assumed that strain energy potential changes only with the first deviatoric invariant. Calculated stress-strain values of generalized Ludwick type material is used by Marlow's model to reproduce stress-strain curve. A commercial finite element analysis software, ABAQUS supports the Marlow strain energy potential form. Plane stress assumption was considered and the geometry was modeled as 2-D shell structure in ABAQUS. The 4-node linear quadrilateral shell elements with reduced integration and hourglass control (S4R) were used to model the mesh geometry.

The effective material properties of a bi-directionally graded FGMs are calculated by [16],

$$E(\bar{x},\bar{y}) = (E_1 - E_2)\left(1 - \frac{\bar{x}}{2L}\right)^{p_x}\left(\frac{1}{2} + \frac{\bar{y}}{h}\right)^{p_y} + E_2, \qquad -h/2 \le \bar{y} \le h/2 \text{ and } 0 \le \bar{x} \le L, \tag{2}$$

where E_1 and E_2 are Young's modulus at certain points and p_x, p_y are the gradation exponents of the FGM (Fig. 3). In the case of bi-directional gradation, Young's modulus of each layers is calculated as follows,

$$E_{i,j} = \frac{1}{\Delta_i \Delta_j}\int_{(i-1)\Delta_i}^{i\Delta_i}\int_{(j-1)\Delta_j}^{j\Delta_j} E(\bar{x},\bar{y})\,d\bar{x}\,d\bar{y}, \tag{3}$$

where i and j represent the order of a layer, Δ_i and Δ_j are the thickness and length of each layer. $E_{i,j}$ is the averaged Young's modulus of a certain layer.

III. Numerical Examples

The first set of numerical examples aimed to examine the generalized Ludwick type bi-directional functionally graded beam by dividing it into a finite number of layers through only thickness direction. To compare the difference between one-directional and two-directional partitions, the beam is modelled as a bi-directionally layered structure in the second set of examples. The cantilever beam in Fig. 1 made of bi-directional functionally graded material obeying generalized Ludwick's constitutive law has a length of L = 1000 mm, a thickness of h = 50 mm, and width of b = 100 mm. The effective material parameters are set to E_1 = 300 MPa, E_2 = 70 MPa, k = 1.5, and ε_0 = 0.07. The gradation exponents are taken to be p_x = 5 and p_y = 2. The only load applied to the beam is the concentrated force F_y having magnitudes of 100, 200, 300, 400, and 500 N and acting towards to vertical direction.

A. Generalized Ludwick Type One-directionally Layered Bi-directional Functionally Graded Beam

The effects of the number of vertical layers on the modelling bi-directional FGM beam are examined in this example. The number of layers through the thickness direction is nt =1, 5, 10, and 20 while it is constant through the length direction as nl = 1. The comparison of deflections and the stress distributions on the beams is given in Fig. 2 in the case that the applied load is 500 N. The maximum tensional normal stresses occur on the top surface of the beam and the magnitudes are 10.8, 8.12, 7.95, and 9.38 MPa for nt = 1, 5, 10, and 20, respectively. However, these maximum values are undoubtedly affected due to the presence of singularity points located at the top and bottom corners of the clamped surface of the beam. Considering Saint - Venant's principle, to avoid the misleading effects of singularity points, stress values are measured at a certain point, where the maximum stress values are converged. The normal stresses measured at that point are 8.89, 6.21, 6.05, and 6.04 MPa for nt = 1, 5, 10, and 20 respectively.

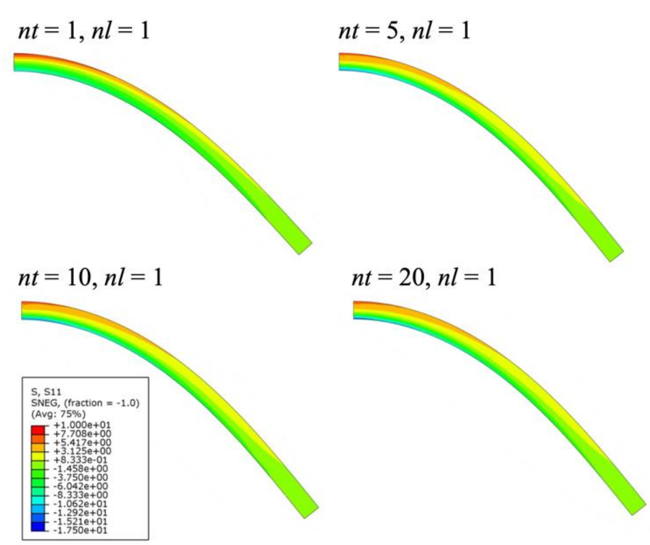

Figure 2: The large deflections and normal stresses on one-directionally layered FGM beam with respect to the number of layers in the thickness direction.

The properly converged normal stress results show that a single layered FGM beam is not capable of representing an actual FGM beam in terms of either stress distributions or large deflections. On the other hand, modeling beams as multi-layered structures are able to simulate the FGM beam accurate enough, regardless of the number of layers.

The vertical (δ_v) and horizontal (δ_h) deflections (in mm) of the beam considering the applied loads are given in Table 1 for each number of layers cases. The maximum vertical and horizontal deflections depend on mostly the applied load. The variation of the number of layers through the thickness direction seems not to affect the deflections significantly. The values for nt = 5, 10, and 20 are very close to each other in this particular example.

Table 1: Deflections of the free end of one directionally layered FGM beam (in mm).

	$nt = 1$ $nl = 1$		$nt = 5$ $nl = 1$		$nt = 10$ $nl = 1$		$nt = 20$ $nl = 1$	
F [N]	δ_v	δ_h	δ_v	δ_h	δ_v	δ_h	δ_v	δ_h
100	144.1	17.8	158.1	19.4	157.5	19.3	157.4	19.2
200	274.7	55.7	300.3	62.9	299.4	62.4	299.2	62.3
300	382.8	104.1	415.2	117.9	414.2	117.1	414.0	117.0
400	468.4	154.9	503.6	174.2	502.6	173.3	502.4	173.1
500	535.2	203.5	570.9	226.9	569.9	225.9	569.8	225.7

B. Generalized Ludwick Type Two-directionally Layered Bi-directional Functionally Graded Beam

The analysis of a two-directionally layered, bi-directional functionally graded beam is presented in this section. The number of layers through the thickness is nt = 20 and through the length is nl = 1, 5, 10, and 20. In Fig. 3, the variation of Young's modulus is given over a bi-directional 20×20 layered FGB. Young's modulus in the left and right top corners are the same and equal to E_2.

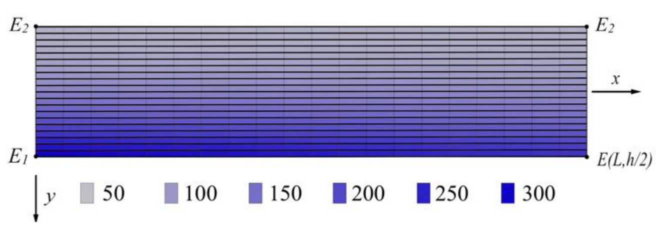

Figure 3: Young's modulus gradient in the bi-directional 20×20 FGB.

The large deflections of the free ends and normal stress distributions on the beams under the effect of concentrated force Fy = 500 N are given in Fig. 4. The converged normal stress values obtained from a specific point far from the singularity regions are 6.04, 5.93, 5.91, and 5.90 MPa for nl = 1, 5, 10, and 20 respectively when nt = 20. The result obtained from a vertically single-layered beam is the most distinctive one with a magnitude of 6.04 MPa. The differences among the multi-layered beams are negligible.

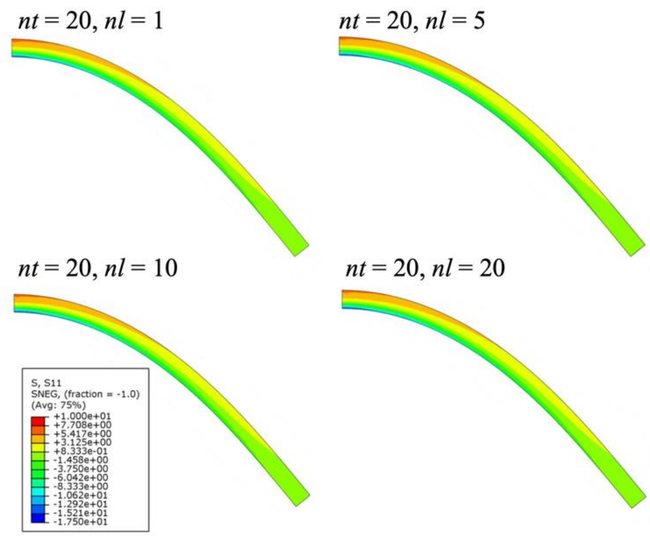

Figure 4: The large deflections and normal stresses of bi-directional FGBs depend on the number of layers in the thickness and length directions.

Table 2 shows the deflections of the end points considering to magnitude of the load and the number of layers. The free end deflection of the 20×1 layered beam is the maximum, while 20×20 layered beam's is the minimum. Besides, the differences in the deflections among two-directionally layered beams are much less than observed between one-directionally layered and the other beams, the same as in the stress distributions.

Table 2: Deflections of the free end of the 2-D FGB (in mm).

	$nt = 20$ $nl = 1$		$nt = 20$ $nl = 5$		$nt = 20$ $nl = 10$		$nt = 20$ $nl = 20$	
F [N]	δ_v	δ_h	δ_v	δ_h	δ_v	δ_h	δ_v	δ_h
100	157.4	19.2	153.5	18.4	153.3	18.4	153.3	18.4
200	299.2	62.3	292.4	59.9	292.2	59.8	292.1	59.8
300	414.0	117.0	405.7	112.8	405.5	112.7	405.4	112.6
400	502.4	173.1	493.7	167.5	493.4	167.4	493.3	167.3
500	569.8	225.7	561.1	219.1	560.9	218.9	560.8	218.9

IV. Conclusion

Evaluation of both material and geometrical nonlinearities rise from the slender geometry; nonlinearly elastic material behavior and functional gradation of effective material properties is an intensely challenging process. Although there are several analytical studies on FGMs, finite element simulations are limited. In this study, a generalized Ludwick type nonlinearly elastic, bi-directional functionally graded beam is modeled as a layered structure to analyze by commercial FEA software. The effects of the number of layers on the normal stress distribution and end deflections are examined. Since the division of the beam into layers is a laborious process and adding more layers directly increases the CPU time, bi-directional FGB is studied as a one-directionally layered structure. Then the same beam is two-directionally divided into layers. Results show that a homogenous beam is not capable of representing an FGB. On the other hand, there is no significant difference between one-directionally layered beam models. Two directionally layered beam models are very successful in simulating bi-directional FGBs. The results are converged at a certain number of layers. The stress distribution and large deflection of FGBs show the almost same behavior in 20×10 and 20×20 layered beams. The results presented here should be verified by further studies to increase our ability to understand the effects of the modelling parameters.

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REFERENCES

[1] A. Candaş, E. Oterkus, and C. E. İmrak, "Peridynamic Simulation of Dynamic Fracture in Functionally Graded Materials Subjected to Impact Load," *Engineering with Computers*, 2021. doi.org/10.1007/s00366-021-01540-2

[2] S. Chucheepsakul, S. Buncharoen, and C. M. Wang, "Large Deflection of Beams under Moment Gradient," *Journal of Engineering Mechanics*, vol. 120(9), pp. 1848–1860, 1994.

[3] C. M. Wang, K. Y. Lam, X. Q. He, and S. Chucheepsakul, "Large Deflections of an End Supported Beam Subjected to a Point Load," *International Journal of Non-Linear Mechanics*, vol. 32(1), pp. 63–72, 1997.

[4] K. Lee, "Large Deflections of Cantilever Beams of Non-Linear Elastic Material under a Combined Loading," *International Journal of Non-Linear Mechanics*, vol. 37(3), pp. 439–443, 2002.

[5] G. Lewis and F. Monasa, "Large Deflections of Cantilever Beams of Nonlinear Materials," *Computers & Structures*, vol. 14(5–6), pp. 357–360, 1981.

[6] G. Lewis and F. Monasa, "Large Deflections of Cantilever Beams of Non-Linear Materials of the Ludwick Type Subjected to an End Moment," *International Journal of Non-Linear Mechanics*, vol. 17(1), pp. 1–6, 1982.

[7] C. Baykara, U. Güven, and I. Bayer, "Large Deflections of a Cantilever Beam of Nonlinear Bimodulus Material Subjected to an End Moment," *Journal of Reinforced Plastics and Composites*, vol. 24(12), pp. 1321–1326, 2005.

[8] J. ho Jung and T. J. Kang, "Large Deflection Analysis of Fibers with Nonlinear Elastic Properties," *Textile Research Journal*, vol. 75(10), pp. 715–723, 2005.

[9] Y. A. Kang, and X. F. Li, "Bending of Functionally Graded Cantilever Beam with Power-Law Non-Linearity Subjected to an End Force," *International Journal of Non-Linear Mechanics*, vol. 44(6), pp. 696–703, 2009.

[10] Y. A. Kang and X. F. Li, "Large Deflections of a Non-Linear Cantilever Functionally Graded Beam," *Journal of Reinforced Plastics and Composites*, vol. 29(12), pp. 1761–1774, 2010.

[11] M. Sitar, F. Kosel, and M. Brojan, "Large Deflections of Nonlinearly Elastic Functionally Graded Composite Beams," *Archives of Civil and Mechanical Engineering*, vol. 14(4), pp. 700–709, 2014.

[12] A. Hacıoğlu, "Large Deflection Analysis for Functionally Graded Cantilever Beams", MSc Thesis, *Graduate School of Science Engineering and Technology, Istanbul Technical University*, pp. 91, 2017.

[13] A. Hacıoğlu and C. Baykara, "A Non-linear Large Deflection Analysis for FGM Beams Under the Different Boundary Conditions and Different Loads Based on Generalized Ludwick Law," in *MECHCOMP3 3rd International Conference on Mechanics of Composites, University of Bologna, Italy,* pp. 9, 4-5 July 2017.

[14] A. Hacıoğlu, A. Candaş, and C. Baykara, "Large deflections of functionally graded non-linearly elastic cantilever beams," *Journal of Engineering Materials and Technology, Transactions of ASME,* (article in press), 2022.

[15] A. Karamanlı, "Elastostatic analysis of two-directional functionally graded beams using various beam theories and Symmetric Smoothed Particle Hydrodynamics method," Composite Structures, vol. 160, pp. 653-669, 2017.

[16] A. Karamanlı, "Bending Analysis of Two Directional Functionally Graded Beams Using a Four-Unknown Shear and Normal Deformation Theory," *Journal of Polytechnic*, vol. 21, 2018. doi.org/10.2339/politeknik.389616.

[17] M. Turan, "Bending analysis of two-directional functionally graded beams using trigonometric series functions", *Archive of Applied Mechanics*, vol. 92, pp. 1841–1858, 2022.

[18] M. Brojan, T. Videnic, F. Kosel, "Large deflections of nonlinearly elastic non-prismatic cantilever beams made from materials obeying the generalized Ludwick constitutive law", *Meccanica,* vol. 44, pp. 733-739, 2009.

[19] R.S. Marlow, "A General First-Invariant Hyperelastic Constitutive Model," *Constitutive Models for Rubber*, pp. 157–160, 2003.