Summary of Structure Equations

n: total number of DOF in the structure
m: unconstrained DOF
k: constrained DOF
$n = m + k$

F is the $n \times 1$ force vector
D is the $n \times 1$ displacement vector
K is the $n \times n$ stiffness matrix
P is the $n \times 1$ fixed end force vector

$$\mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \\ \\ \ldots \\ \\ F_n \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} D_1 \\ D_2 \\ \\ \ldots \\ \\ D_n \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} P_1 \\ P_2 \\ \\ \ldots \\ \\ P_n \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} K_{11} & K_{12} & \ldots & K_{1n} \\ K_{21} & K_{22} & & \\ & & & \\ \ldots & & \ldots & \\ & & & \\ K_{n1} & & & K_{nn} \end{bmatrix}$$

$$\mathbf{F} = \mathbf{KD} + \mathbf{P}$$

$$\begin{bmatrix} F_1 \\ F_2 \\ \\ \ldots \\ \\ F_n \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & \ldots & K_{1n} \\ K_{21} & K_{22} & & \\ & & & \\ \ldots & & \ldots & \\ & & & \\ K_{n1} & & & K_{nn} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ \\ \ldots \\ \\ D_n \end{bmatrix} + \begin{bmatrix} P_1 \\ P_2 \\ \\ \ldots \\ \\ P_n \end{bmatrix}$$

$$F_1 = K_{11}D_1 + K_{12}D_2 + \ldots + K_{1n}D_n + P_1$$
$$F_2 = K_{21}D_1 + K_{22}D_2 + \ldots + K_{2n}D_n + P_2$$
$$\ldots$$
$$F_n = K_{n1}D_1 + K_{n2}D_2 + \ldots + K_{nn}D_n + P_n$$

Partitioning

$$\begin{array}{ccccccc} & & m\text{ - columns} \quad\quad k\text{ - columns} & & & & \\ \begin{bmatrix} F_1 \\ F_2 \\ \hline \cdots \\ F_n \end{bmatrix} & = & \left[\begin{array}{cc|cc} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & & \\ \hline \cdots & & \cdots & \\ K_{n1} & & & K_{nn} \end{array}\right] & \begin{bmatrix} D_1 \\ D_2 \\ \hline \cdots \\ D_n \end{bmatrix} & + & \begin{bmatrix} P_1 \\ P_2 \\ \hline \cdots \\ P_n \end{bmatrix} & \begin{array}{l} m\text{ - rows} \\ \\ k\text{ - rows} \\ \end{array} \end{array}$$

$$\begin{array}{ccccccc} & & m\text{ - columns} \quad k\text{ - columns} & & & & \\ \begin{bmatrix} \overset{\text{known}}{\mathbf{F}_\mathrm{A}} \\ \hline \underset{\text{unknown}}{\mathbf{F}_B} \end{bmatrix} & = & \underset{\text{known}}{\left[\begin{array}{c|c} \mathbf{K}_\mathrm{AA} & \mathbf{K}_\mathrm{AB} \\ \hline \mathbf{K}_\mathrm{BA} & \mathbf{K}_\mathrm{BB} \end{array}\right]} & \begin{bmatrix} \overset{\text{unknown}}{\mathbf{D}_\mathrm{A}} \\ \hline \underset{\text{known}}{\mathbf{D}_\mathrm{B}} \end{bmatrix} & + & \begin{bmatrix} \overset{\text{known}}{\mathbf{P}_\mathrm{A}} \\ \hline \cdots \\ \mathbf{P}_\mathrm{B} \end{bmatrix} & \begin{array}{l} m\text{ - rows} \\ \\ k\text{ - rows} \end{array} \end{array}$$

$$\begin{bmatrix} \mathbf{F}_\mathrm{A} \\ \mathbf{F}_\mathrm{B} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_\mathrm{AA} & \mathbf{K}_\mathrm{AB} \\ \mathbf{K}_\mathrm{BA} & \mathbf{K}_\mathrm{BB} \end{bmatrix} \begin{bmatrix} \mathbf{D}_\mathrm{A} \\ \mathbf{D}_B \end{bmatrix} + \begin{bmatrix} \mathbf{P}_\mathrm{A} \\ \mathbf{P}_\mathrm{B} \end{bmatrix}$$

$$\mathbf{F}_\mathrm{A} = \mathbf{K}_\mathrm{AA}\mathbf{D}_\mathrm{A} + \mathbf{K}_\mathrm{AB}\mathbf{D}_\mathrm{B} + \mathbf{P}_\mathrm{A} \qquad (1)$$

$$\mathbf{F}_\mathrm{B} = \mathbf{K}_\mathrm{BA}\mathbf{D}_\mathrm{A} + \mathbf{K}_\mathrm{BB}\mathbf{D}_\mathrm{B} + \mathbf{P}_\mathrm{B} \qquad (2)$$

Note that when there is no support settlement and support rotation, $\mathbf{D}_\mathrm{B} = \mathbf{0}$ (displacement and rotations at unconstrained DOF).

Solution of the Equations

(1) gives $\mathbf{D}_A = \mathbf{K}_{AA}^{-1}(\mathbf{F}_A - \mathbf{K}_{AB}\mathbf{D}_B - \mathbf{P}_A)$ (displacements and rotations at unconstrained DOF)

(2) gives $\mathbf{F}_B = \mathbf{K}_{BA}\mathbf{D}_A + \mathbf{K}_{BB}\mathbf{D}_B + \mathbf{P}_B$ (support reactions - reactions at constrained DOF)

Member end displacements in global coordinates, **d**, can be found from **D**.

Member end displacements in local coordinates and member end forces in local and global coordinates can be found using **d**.