MAK 411 E
Experimental Methods in Engineering

"Review of Experimental Methods

&

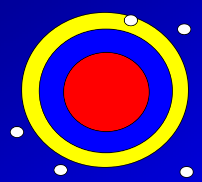
Uncertainty and Error Analysis"

Basic Terminology

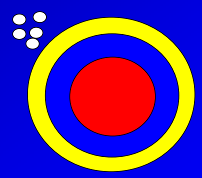
Term Confusion:

Accuracy, Precision, Resolution, Sensitivity, Instability, Noise, Repeatability, Reliability, Dispersion, etc...

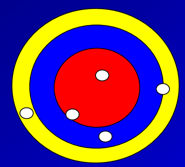
The bull's-eye (●) stands for the "true value"



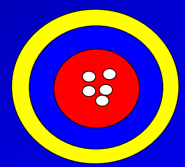
Shots are widely distributed and mostly off-target.
Inaccurate, Unrepeatable



Precise but Inaccurate



Accurate; two in the "bull"
But the shots are dispersed:
Imprecise



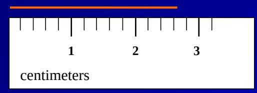
Both Accurate and Precise
Little deviation from "true"

Does the attribute **Accuracy** always implies **Precision**?

Not necessarily!
An inaccurate device may be malfunctioning precisely.

Resolution is mistaken to be the same as **Accuracy**

<u>**Resolution**</u> is just the discrimination that the instrument can show



The resolution of the ruler is 2 mm (one fifth of a centimeter).

2.6 ± 0.2 cm.

2.65 ± 0.05 cm.

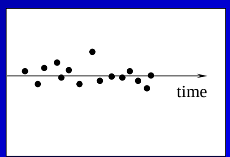
What is the difference between **Sensitivity** and **Resolution**?

Resolution is a measure of the smallest change in output (indication) that is possible sensitivity relates to the smallest change in the input (stimulus) that causes a discernible change in the output. So there *is* an association between these two terms.

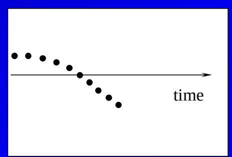
Instability and **Noise** are both qualifications of change over time.

Stability is commonly used for changes over periods of a second or more.

Noise, on the other hand, is used for changes over shorter intervals.



Noise



Instability

Error Analysis

The knowledge we have of the physical world is obtained by doing experiments and making measurements. It is important to understand how to express such data and how to analyze and draw meaningful conclusions from it.

In doing this it is crucial to understand that all measurements of physical quantities are subject to uncertainties. It is never possible to measure anything exactly. It is good, of course, to make the error as small as possible but it is always there. And in order to draw valid conclusions the error must be indicated and dealt with properly.

Example:

Person's height = 176.5 cm.

The height of a person depends on whether her shows are on, whether her hair is long and how it is made up. These factors are called "*errors of definition*".

Even if someone defines the circumstances, the result will still have an error associated with it. The scale has a limited **resolution**.

A measurement is not a "measured value" only. It must contain a measured value and an estimated value.

Rules to remember

Significant Figures: The significant figures of a (measured or calculated) quantity are the **meaningful** digits in it. There are conventions which you should learn and follow for how to express numbers so as to properly indicate their significant figures.

- Any digit that is not zero is significant. Thus 549 has three significant figures and 1.892 has four significant figures.
- Zeros between non zero digits are significant. Thus 4023 has four significant figures.
- Zeros to the left of the first non zero digit are not significant. Thus 0.000034 has only two significant figures. This is more easily seen if it is written as 3.4×10^{-5}.
- For numbers with decimal points, zeros to the right of a non zero digit are significant. Thus 2.00 has three significant figures and 0.050 has two significant figures. For this reason it is important to keep the trailing zeros to indicate the actual number of significant figures.

- For numbers without decimal points, trailing zeros may or may not be significant. Thus, 400 indicates only one significant figure. To indicate that the trailing zeros are significant a decimal point must be added. For example, 400. has three significant figures, and has one significant figure. Exact numbers have an infinite number of significant digits. For example, if there are two oranges on a table, then the number of oranges is 2.000... . Defined numbers are also like this. For example, the number of centimeters per inch (2.54) has an infinite number of significant digits, as does the speed of light (299792458 m/s).

There are also specific rules for how to consistently express the uncertainty associated with a number. In general, the last significant figure in any result should be of the same order of magnitude (i.e.. in the same decimal position) as the uncertainty. Also, the uncertainty should be rounded to one or two significant figures. Always work out the uncertainty after finding the number of significant figures for the actual measurement.

For example,
9.82 ± 0.02
10.0 ± 1.5
4 ± 1

The following numbers are all incorrect.
9.82 ± 0.02385 is wrong but 9.82 ± 0.02 is fine
10.0 ± 2 is wrong but 10.0 ± 2.0 is fine
4 ± 0.5 is wrong but 4.0 ± 0.5 is fine

In practice, when doing mathematical calculations, it is a good idea to keep one more digit than is significant to reduce rounding errors. But in the end, the answer must be expressed with only the proper number of significant figures. After addition or subtraction, the result is significant only to the **place** determined by the largest last significant place in the original numbers.

For example,
89.332 + 1.1 = 90.432
should be rounded to get 90.4 (the tenths place is the last significant place in 1.1). After multiplication or division, the **number** of significant figures in the result is determined by the original number with the smallest number of significant figures.

For example,
(2.80) (4.5039) = 12.61092
should be rounded off to 12.6 (three significant figures like 2.80).

Error

Error does not mean a difference between what is expected to be found and the actually measured value.

Error does not mean "blunder"

Error, has to do with uncertainty in measurements that nothing can be done about. If a measurement is repeated, the values obtained will differ and none of the results can be preferred over the others. Although it is not possible to do anything about such error, it can be characterized. For instance, the repeated measurements may cluster tightly together or they may spread widely. This pattern can be analyzed systematically.

Classsification of Error

- Systematic
- Random

Systematic errors are errors which tend to shift all measurements in a systematic way so their mean value is displaced. This may be due to such things as incorrect calibration of equipment, consistently improper use of equipment or failure to properly account for some effect. In a sense, a systematic error is rather like a blunder and large systematic errors can and must be eliminated in a good experiment. But small systematic errors will always be present. For instance, no instrument can ever be calibrated perfectly.

Random errors are errors which fluctuate from one measurement to the next. They yield results distributed about some mean value. They can occur for a variety of reasons.

- They may occur due to lack of sensitivity. For a sufficiently a small change an instrument may not be able to respond to it or to indicate it or the observer may not be able to discern it.
- They may occur due to noise. There may be extraneous disturbances which cannot be taken into account.
- They may be due to imprecise definition.
- They may also occur due to statistical processes such as the roll of dice.

Random errors displace measurements in an arbitrary direction whereas systematic errors displace measurements in a single direction. Some systematic error can be substantially eliminated (or properly taken into account). Random errors are unavoidable and must be lived with.

Many times you will find results quoted with two errors. The first error quoted is usually the random error, and the second is called the systematic error. If only one error is quoted, then the errors from all sources are added together.

A good example of "random error" is the statistical error associated with sampling or counting. For example, consider radioactive decay which occurs randomly at a some (average) rate. If a sample has, on average, 1000 radioactive decays per second then the expected number of decays in 5 seconds would be 5000. A particular measurement in a 5 second interval will, of course, vary from this average but it will generally yield a value **within** 5000 ± . Behavior like this, where the error,

$$\Delta n = \sqrt{\Delta n_{expected}} \quad . (1)$$

is called a Poisson statistical process. Typically if one does not know $n_{expected}$ it is assumed that,

$$n_{expected} = n_{measured}$$

in order to estimate this error.

Mean Value

$$x_1, x_2, \ldots, x_k, \ldots, x_N$$

$$\bar{x} = \frac{x_1 + x_2 + \ldots + x_k + \ldots + x_N}{N} = \frac{\sum_{k=1}^{N} x_k}{N}$$

Measuring Error

There are several different ways the distribution of the measured values of a repeated experiment such as discussed above can be specified.

•Maximum Error

The maximum and minimum values of the data set, x_{max} and x_{min}, could be specified. In these terms, the quantity,

$$\Delta x_{max} = \frac{x_{max} - x_{min}}{2}$$

is the maximum error. And virtually no measurements should ever fall outside $\bar{x} \pm \Delta x_{max}$

•Probable Error

The probable error, Δx_{prob}, specifies the range $\bar{x} \pm \Delta x_{prob}$ which contains 50% of the measured values.

•Average Deviation

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$$\Delta x_{ad} = \frac{\sum_k |x_k - \bar{x}|}{N}$$

For a Gaussian distribution of the data, about 58% will lie within $\bar{x} \pm \Delta x_{ad}$

•Standard Deviation

The mean is the most probable value of a Gaussian distribution. In terms of the mean, **the standard deviation** of any distribution is,

$$\sigma = \sqrt{\frac{\sum_k (x_k - \bar{x})^2}{N}}$$

The quantity, the square of the standard deviation, is called the **variance**. The best estimate of the standard deviation is **the sample standard deviation** (also shown by S_x)

$$\sigma_x = \sqrt{\frac{\sum_k (x_k - \bar{x})^2}{N-1}}$$

•Error of the mean:

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}} = \sqrt{\frac{\sum_k (x_k - \bar{x})^2}{N(N-1)}}$$

If a measurement is made there is 68% probability that the measurement will lie within the $\bar{x} \pm \sigma_{\bar{x}}$

Example

Suppose the number of cosmic ray particles passing through some detecting device every hour is measured nine times and the results are those in the following table.

Thus we have = 900/9 = 100 and = 1500/8 = 188 or = 14. Then the probability that one more measurement of **x** will lie within 100 ± 14 is 68%.
The value to be reported for this series of measurements is 100±(14/3) or 100 ± 5. If one were to make another series of nine measurements of **x** there would be a 68% probability the new mean would lie within the range 100 ± 5.

i	x_i	$(x_i\text{-}x_m)^2$
1	80	400
2	95	25
3	100	0
4	110	100
5	90	100
6	115	225
7	85	225
8	120	400
9	105	25
S	900	1500

Propagation of Errors

Frequently, the result of an experiment will not be measured directly. Rather, it will be calculated from several measured physical quantities (each of which has a mean value and an error). What is the resulting error in the final result of such an experiment?
For instance, what is the error in **Z = A + B** where **A** and **B** are two measured quantities with errors and respectively?
A first thought might be that the error in **Z** would be just the sum of the errors in **A** and **B**. After all,

$$(A + \Delta A) + (B + \Delta B) = (A + B) + (\Delta A + \Delta B)$$

and

$$(A - \Delta A) + (B - \Delta B) = (A + B) - (\Delta A + \Delta B)$$

But this assumes that, when combined, the errors in **A** and **B** have the same sign and maximum magnitude; that is that they always combine in the worst possible way. This could only happen if the errors in the two variables were perfectly correlated, (i.e.. if the two variables were not really independent).
If the variables are independent then sometimes the error in one variable will happen to cancel out some of the error in the other and so, on the average, the error in **Z** will be less than the sum of the errors in its parts.

A reasonable way to try to take this into account is to treat the perturbations in **Z** produced by perturbations in its parts as if they were "perpendicular" and added according to the Pythagorean theorem,

$$\Delta Z = \sqrt{(\Delta A)^2 + (\Delta B)^2}$$

That is, if **A** = (100 ± 3) and **B** = (6 ± 4) then **Z** = (106 ± 5) since $5 = \sqrt{3^2 + 4^2}$

This idea can be used to derive a general rule. Suppose there are two measurements, A and B, and the final result is **Z = F(A, B)** for some function **F**. If **A** is perturbed by ΔA then **Z** will be perturbed by

$$\left(\frac{\partial F}{\partial A}\right)\Delta A$$

where (the partial derivative) $\partial F / \partial A$ is the derivative of **F** with respect to **A** with **B** held constant. Similarly the perturbation in **Z** due to a perturbation in **B** is,

$$\left(\frac{\partial F}{\partial B}\right)\Delta B$$

Combining these by the Pythagorean theorem yields

$$\Delta Z = \sqrt{\left(\frac{\partial F}{\partial A}\right)^2 (\Delta A)^2 + \left(\frac{\partial F}{\partial B}\right)^2 (\Delta B)^2}$$

In the example of Z = A + B considered above, $\frac{\partial F}{\partial A} = 1$ and $\frac{\partial F}{\partial B} = 1$

so this gives the same result as before.

Similarly if Z = A - B then, $\frac{\partial F}{\partial A} = 1$ and $\frac{\partial F}{\partial B} = -1$

which also gives the same result. Errors combine in the same way for both addition and subtraction

However, if Z = AB then, $\frac{\partial F}{\partial A} = B$ and $\frac{\partial F}{\partial B} = A$

$$\Delta Z = \sqrt{B^2 (\Delta A)^2 + A^2 (\Delta B)^2}$$

$$\frac{\Delta Z}{z} = \frac{\Delta Z}{AB} = \sqrt{(\frac{\Delta A}{A})^2 + (\frac{\Delta B}{B})^2}$$

or the fractional error in **Z** is the square root of the sum of the squares of the fractional errors in its parts.