ADDITIONAL HOMEWORK 4

58)

Use Cauchy's residue theorem to evaluate the integral of each of these functions around the circle $|z| = 3$ in the positive sense:

(a) $\dfrac{\exp(-z)}{z^2}$; (b) $\dfrac{\exp(-z)}{(z-1)^2}$; (c) $z^2 \exp\left(\dfrac{1}{z}\right)$; (d) $\dfrac{z+1}{z^2-2z}$.

Ans. (a) $-2\pi i$; (b) $-2\pi i/e$; (c) $\pi i/3$; (d) $2\pi i$.

59)

Show that

(a) $\displaystyle \operatorname*{Res}_{z=-1} \frac{z^{1/4}}{z+1} = \frac{1+i}{\sqrt{2}}$ $\quad (|z| > 0,\ 0 < \arg z < 2\pi)$;

(b) $\displaystyle \operatorname*{Res}_{z=i} \frac{\operatorname{Log} z}{(z^2+1)^2} = \frac{\pi + 2i}{8}$;

(c) $\displaystyle \operatorname*{Res}_{z=i} \frac{z^{1/2}}{(z^2+1)^2} = \frac{1-i}{8\sqrt{2}}$ $\quad (|z| > 0,\ 0 < \arg z < 2\pi)$.

60)

Find the value of the integral

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}\, dz,$$

taken counterclockwise around the circle (a) $|z-2| = 2$; (b) $|z| = 4$.

Ans. (a) πi; (b) $6\pi i$.

61)

Use residues to evaluate the improper integrals in Exercises 1 through 5.

1. $\displaystyle \int_0^\infty \frac{dx}{x^2+1}$.

   Ans. $\pi/2$.

2. $\displaystyle \int_0^\infty \frac{dx}{(x^2+1)^2}$.

   Ans. $\pi/4$.

3. $\displaystyle \int_0^\infty \frac{dx}{x^4+1}$.

   Ans. $\pi/(2\sqrt{2})$.

4. $\displaystyle \int_0^\infty \frac{x^2\, dx}{(x^2+1)(x^2+4)}$.

   Ans. $\pi/6$.

5. $\displaystyle \int_0^\infty \frac{x^2\, dx}{(x^2+9)(x^2+4)^2}$.

   Ans. $\pi/200$.

62)

Use residues to evaluate the improper integrals in Exercises 1 through 8.

1. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+a^2)(x^2+b^2)}$ $(a > b > 0)$.

Ans. $\displaystyle\frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$.

2. $\displaystyle\int_0^{\infty} \frac{\cos\, ax}{x^2+1}\, dx$ $(a > 0)$.

Ans. $\displaystyle\frac{\pi}{2} e^{-a}$.

3. $\displaystyle\int_0^{\infty} \frac{\cos\, ax}{(x^2+b^2)^2}\, dx$ $(a > 0, b > 0)$.

Ans. $\displaystyle\frac{\pi}{4b^3}(1+ab)e^{-ab}$.

4. $\displaystyle\int_0^{\infty} \frac{x \sin 2x}{x^2+3}\, dx$.

Ans. $\displaystyle\frac{\pi}{2}\exp(-2\sqrt{3})$.

5. $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin ax}{x^4+4}\, dx$ $(a > 0)$.

Ans. $\displaystyle\frac{\pi}{2} e^{-a}\sin a$.

6. $\displaystyle\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4+4}\, dx$ $(a > 0)$.

Ans. $\pi e^{-a}\cos a$.

7. $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin x\, dx}{(x^2+1)(x^2+4)}$.

8. $\displaystyle\int_0^{\infty} \frac{x^3 \sin x\, dx}{(x^2+1)(x^2+9)}$.

63)

Use residues to evaluate the definite integrals in Exercises 1 through 7.

1. $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta}$.

Ans. $\dfrac{2\pi}{3}$.

2. $\displaystyle\int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2\theta}$.

Ans. $\sqrt{2}\pi$.

3. $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta \, d\theta}{5 - 4\cos 2\theta}$.

Ans. $\dfrac{3\pi}{8}$.

4. $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta}$ $\quad (-1 < a < 1)$.

Ans. $\dfrac{2\pi}{\sqrt{1 - a^2}}$.

5. $\displaystyle\int_0^{\pi} \frac{\cos 2\theta \, d\theta}{1 - 2a\cos\theta + a^2}$ $\quad (-1 < a < 1)$.

Ans. $\dfrac{a^2\pi}{1 - a^2}$.

6. $\displaystyle\int_0^{\pi} \frac{d\theta}{(a + \cos\theta)^2}$ $\quad (a > 1)$.

Ans. $\dfrac{a\pi}{(\sqrt{a^2 - 1})^3}$.

7. $\displaystyle\int_0^{\pi} \sin^{2n}\theta \, d\theta$ $\quad (n = 1, 2, \ldots)$.

Ans. $\dfrac{(2n)!}{2^{2n}(n!)^2}\pi$.

64)

Find the residue at $z = 0$ of the function $f(z) = ze^{3/z}$ and compute

$$\oint_{|z|=4} ze^{3/z}\, dz.$$

65)

Compute the residues at the singularities of

$$f(z) = \frac{\cos z}{z^2(z - \pi)^3}.$$

66)

Evaluate

$$\oint_{|z|=2} \frac{1-2z}{z(z-1)(z-3)}\,dz.$$

67)

Compute

$$\oint_{|z|=5} \left[ze^{3/z} + \frac{\cos z}{z^2(z-\pi)^3}\right] dz.$$

68)

Determine all the isolated singularities of each of the following functions and compute the residue at each singularity.

(a) $\dfrac{e^{3z}}{z-2}$ (b) $\dfrac{z+1}{z^2-3z+2}$ (c) $\dfrac{\cos z}{z^2}$ (d) $\left(\dfrac{z-1}{z+1}\right)^3$

(e) $\dfrac{e^z}{z(z+1)^3}$ (f) $\sin\left(\dfrac{1}{3z}\right)$ (g) $\tan z$ (h) $\dfrac{z-1}{\sin z}$

(i) $z^2/(1-\sqrt{z})$, where $\sqrt{z}$ denotes the principal branch.