ADDITIONAL HOMEWORK 3

41)

Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals:

(a) $\int_C \frac{e^{-z}\,dz}{z - (\pi i/2)}$; (b) $\int_C \frac{\cos z}{z(z^2+8)}\,dz$; (c) $\int_C \frac{z\,dz}{2z+1}$;

(d) $\int_C \frac{\cosh z}{z^4}\,dz$; (e) $\int_C \frac{\tan(z/2)}{(z-x_0)^2}\,dz$ $(-2 < x_0 < 2)$.

Ans. (a) 2π; (b) $\pi i/4$; (c) $-\pi i/2$; (d) 0; (e) $i\pi \sec^2(x_0/2)$.

42)

Find the value of the integral of $g(z)$ around the circle $|z - i| = 2$ in the positive sense when

(a) $g(z) = \frac{1}{z^2+4}$; (b) $g(z) = \frac{1}{(z^2+4)^2}$.

Ans. (a) $\pi/2$; (b) $\pi/16$.

43)

Let C be the circle $|z| = 3$, described in the positive sense. Show that if

$$g(w) = \int_C \frac{2z^2 - z - 2}{z - w}\,dz \qquad (|w| \neq 3),$$

then $g(2) = 8\pi i$. What is the value of $g(w)$ when $|w| > 3$?

44)

Let C be any simple closed contour, described in the positive sense in the z plane, and write

$$g(w) = \int_C \frac{z^3 + 2z}{(z-w)^3}\,dz.$$

Show that $g(w) = 6\pi i w$ when w is inside C and that $g(w) = 0$ when w is outside C.

45)

Show that if f is analytic within and on a simple closed contour C and z_0 is not on C, then

$$\int_C \frac{f'(z)\,dz}{z - z_0} = \int_C \frac{f(z)\,dz}{(z-z_0)^2}.$$

46)

Evaluate

$$\oint_{|z|=2} \frac{e^z}{z^2-9}\,dz.$$

47)

Determine the domain of analyticity for each of the given functions f and explain why

$$\oint_{|z|=2} f(z)\,dz = 0.$$

(a) $f(z) = \dfrac{z}{z^2+25}$

(b) $f(z) = e^{-z}(2z+1)$

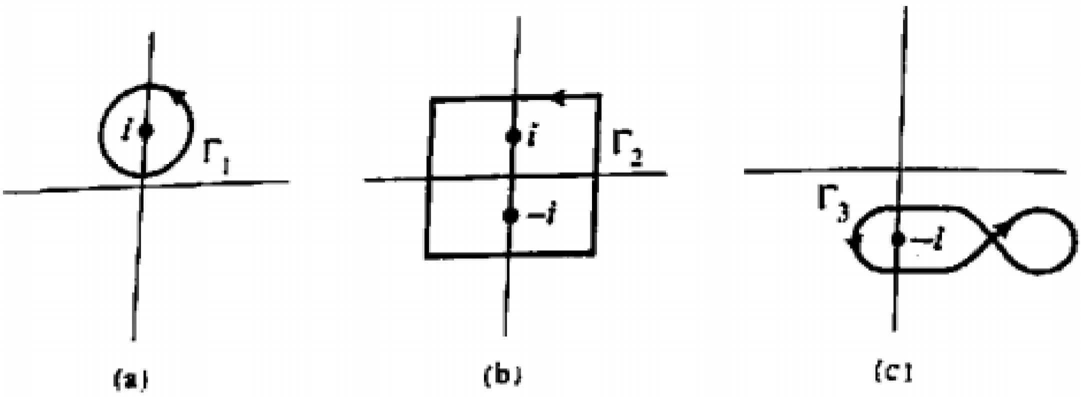
(c) $f(z) = \dfrac{\cos z}{z^2-6z+10}$

(d) $f(z) = \operatorname{Log}(z+3)$

(e) $f(z) = \sec\left(\dfrac{z}{2}\right)$

48)

Evaluate $\int 1/(z^2+1)\,dz$ along the three closed contours Γ_1, Γ_2, Γ_3 in Fig.



49)

Let C be the circle $|z| = 2$ traversed once in the positive sense. Compute each of the following integrals.

(a) $\displaystyle\int_C \frac{\sin 3z}{z-\frac{\pi}{2}}\,dz$

(b) $\displaystyle\int_C \frac{ze^z}{2z-3}\,dz$

(c) $\displaystyle\int_C \frac{\cos z}{z^3+9z}\,dz$

(d) $\displaystyle\int_C \frac{5z^2+2z+1}{(z-i)^3}\,dz$

(e) $\displaystyle\int_C \frac{e^{-z}}{(z+1)^2}\,dz$

(f) $\displaystyle\int_C \frac{\sin z}{z^2(z-4)}\,dz$

50)

Compute

$$\int_C \frac{z+i}{z^3+2z^2}\,dz,$$

where C is

(a) the circle $|z| = 1$ traversed once counterclockwise.

(b) the circle $|z+2-i| = 2$ traversed once counterclockwise.

(c) the circle $|z-2i| = 1$ traversed once counterclockwise.

51)

Evaluate

$$\int_\Gamma \frac{e^{iz}}{(z^2+1)^2}\,dz,$$

where Γ is the circle $|z| = 3$ traversed once counterclockwise. [HINT: Show that the integral can be written as the sum of two integrals around small circles centered at the singularities.]

52)

Evaluate each of the following integrals.

(a) $\displaystyle\int_1^i (z+1)^4 dz$

(b) $\displaystyle\int_0^{\pi i} e^z\,dz$

(c) $\displaystyle\int_{-i}^i \cosh 2z\,dz$

(d) $\displaystyle\int_C \frac{e^{2z}}{z^2+1}\,dz,\ C:\ z-1 = e^{it},\ 0 \le t \le 2\pi$

(e) $\displaystyle\int_{C^+} \frac{dz}{z^2-1}$, where C^+ is any simple closed contour enclosing both $z = 1$ and $z = -1$.

53)

If C: $z = i + 3e^{it}$, $0 \le t \le 2\pi$, evaluate each of the following integrals.

(a) $\displaystyle\int_C \frac{2z^2-z+3}{z+1}\,dz$

(b) $\displaystyle\int_C \frac{e^{-z}}{z}\,dz$

(c) $\displaystyle\int_C \frac{\cos 2\pi z}{3z-1}\,dz$

(d) $\displaystyle\int_C \frac{\sin 2z}{2z-i}\,dz$

(e) $\displaystyle\int_C \frac{(z+1)\,dz}{(z^2-4)(z+4)}$

(f) $\displaystyle\int_C \frac{\sin z}{z^2-2z}\,dz$

54)

. If C: $z = 2e^{it}$, $0 \leq t \leq 4\pi$, evaluate each of the following:

(a) $\displaystyle\int_C \frac{z^2}{z^4 - 1}\,dz$ (b) $\displaystyle\int_C \frac{z^3 + z + 3}{z - i}\,dz$

(c) $\displaystyle\int_C \frac{e^z \cos z}{(z - i\pi)(z - {}^1\!/_2 i\pi)}\,dz$ (d) $\displaystyle\int_C \frac{e^{z-1}\,dz}{z(z^2 - 1)}$

55)

Show that the substitution $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$, transforms the real integral

$$\int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta} \qquad (1)$$

into the complex integral

$$-i \int_C \frac{dz}{2z^2 + 5z + 2} \qquad (2)$$

where C: $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$, and use (2) to evaluate (1).

56)

Evaluate each of the following integrals.

(a) $\displaystyle\int_C \frac{z^4 + z^2 + 3z + 1}{(z - i)^3}\,dz$, C: $z = 2e^{it}, 0 \leq t \leq 2\pi$

(b) $\displaystyle\int_C \frac{e^{2z}}{z^5}\,dz$, C: $z = e^{it}$, $0 \leq t \leq 2\pi$

(c) $\displaystyle\int_C \frac{z^5}{(z - 1)^6}\,dz$, C: $z = 2e^{it}$, $0 \leq t \leq 6\pi$

(d) $\displaystyle\int_C \frac{\cosh 4z}{z^3}\,dz$, C: $z = e^{-it}$, $0 \leq t \leq 4\pi$

(e) $\displaystyle\int_C \frac{\sin z}{(z^2 + 1)^2}\,dz$, C: $z = 1 + 2e^{it}$, $0 \leq t \leq 2\pi$

57)

Find the residue at $z = 0$ of the function

(*a*) $\dfrac{1}{z + z^2}$; (*b*) $z\cos\left(\dfrac{1}{z}\right)$; (*c*) $\dfrac{z - \sin z}{z}$; (*d*) $\dfrac{\cot z}{z^4}$; (*e*) $\dfrac{\sinh z}{z^4(1 - z^2)}$.

Ans. (*a*) 1; (*b*) $-1/2$; (*c*) 0; (*d*) $-1/45$; (*e*) $7/6$.