ADDITIONAL HOMEWORK 1

2)

Show that:

(a) $i^3 = -i$ (b) $i^4 = 1$

(c) $i^{4n} = 1,\ i^{4n+1} = i,\ i^{4n+2} = -1,\ i^{4n+3} = -i$

(d) $\dfrac{1+i+i^3+i^5}{1+i^2+i^4+i^6+i^8} = 1+i$

4)

Describe graphically the sets of points in the complex plane defined by the following equalities or inequalities.

1. $\operatorname{Re} z \geq 0$
2. $\operatorname{Im} z < 0$
3. $|z| = 2$
4. $|z-1| < 1$
5. $z = 3 + (1+i)t,\ t \in \mathbb{R}$
6. $z = (1-t)i + t,\ 0 \leq t \leq 1$
7. $(z-i)(\bar{z}+i) = 4$
8. $z + 1 = 3e^{it},\ 0 \leq t \leq 2\pi$
9. $\operatorname{Arg} z = \dfrac{\pi}{4}$
10. $\operatorname{Im} \dfrac{(z-2)}{i} > 0$
11. $\dfrac{-\pi}{2} < \operatorname{Arg} z < \dfrac{\pi}{4}$
12. $0 < \operatorname{Arg} z < \dfrac{3\pi}{4}$
13. $\left\{z\colon\ |z| > 1, |\operatorname{Arg} z| \leq \dfrac{\pi}{4}\right\}$
14. $\operatorname{Re}\left(\dfrac{1}{z}\right) = 2$
15. $|z+1| = |z-2|$
16. $a < \operatorname{Re} z < b,\ a, b$ real
17. $2 \leq |z-i| \leq 3$
18. $|z-1| \leq |z+1|$
19. $|z-1| \leq 2|z+1|$
20. $|z| - \operatorname{Re} z \leq {}^1\!/_2$

5)

Show that:

(a) $\lim\limits_{z\to i}\left[3y^2 + i\dfrac{\tan(y-1)}{y-1}\right] = 3+i$

(b) $\lim\limits_{z\to 2-i} \bar{z} = 2+i$

(c) $\lim\limits_{z\to\infty} \dfrac{z+i}{z-1} = 1$

(d) $\lim\limits_{z\to i} \dfrac{1}{z^2+1} = \infty$

(e) $\lim\limits_{z\to\infty} (z^3+1) = \infty$

7)

Discuss the continuity and uniform continuity of the following functions on the domains given.

(a) $f(z) = z^3,\ D = \{z\colon\ |z| < 2\}$

(b) $f(z) = \dfrac{1}{z+i},\ D = \{z\colon\ |z| < 1\}$

(c) $f(z) = \dfrac{z+1}{z},\ D = \{z\colon\ |z| > 1\}$

8)

Find the derivatives of the following functions.

(a) $w = z^n$ (n a positive integer) (b) $w = z^{-n}$

(c) $w = e^{az}$ (d) $w = \sin z$

(e) $w = \cos z$ (f) $w = \tan z$

(g) $w = \sinh z$ (h) $w = \cosh z$

(i) $w = \tanh z$ (j) $w = {}^1\!/_5 e^z(\sin^2 z - \sin 2z + 2)$

9)

Let $z = re^{i\theta}$ and $f(z) = u(r,\theta) + iv(r,\theta)$ for $z \in D$. Suppose that u and v are differentiable in some open set $A \subset D$ that does not contain the point $z = 0$.

(a) Prove that f is analytic in A iff

$$ru_r = v_\theta \qquad \text{and} \qquad rv_r = -u_\theta$$

at every point of A (Cauchy–Riemann equations in polar coordinates).

(b) Show that if $f'(z)$ exists at $z \in A$, then $f'(z) = e^{-i\theta}(u_r + iv_r)$

(c) Assuming the existence and continuity of $u_{r\theta}$ and $v_{r\theta}$ in A, prove that both u and v satisfy in A the Laplace equation in polar form, namely,

$$r^2 \frac{\partial^2 \psi}{\partial r^2} + r\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial \theta^2} = 0$$

10)

Show that each of the following functions is harmonic in some domain, then determine the corresponding harmonic conjugate and $f(z) = u+iv$ as a function of z.

(a) $u = x^4 - 6x^2y^2 + y^4$

(b) $v = 4x^3y - 4xy^3 + 2xy$

(c) $u = \dfrac{x}{x^2 + y^2}$

(d) $v = \operatorname{Arc\,tan} \dfrac{y}{x}$

(e) $u = e^x(x \cos y - y \sin y)$

(f) $v = e^{-x}(y \cos y - x \sin y)$

(g) $u = \sin x \cosh y$

12)

Find the principal value of

(a) i^i; (b) $\left[\frac{e}{2}(-1 - \sqrt{3}i)\right]^{3\pi i}$; (c) $(1 - i)^{4i}$.

Ans. (a) $\exp(-\pi/2)$; (b) $-\exp(2\pi^2)$; (c) $e^{\pi}[\cos(2 \ln 2) + i \sin(2 \ln 2)]$.

15)

Write the function $f(z) = z^3 + z + 1$ in the form $f(z) = u(x, y) + iv(x, y)$.

17)

show that $f'(z)$ does not exist at any point if

(a) $f(z) = \bar{z}$; (b) $f(z) = z - \bar{z}$; (c) $f(z) = 2x + ixy^2$; (d) $f(z) = e^x e^{-iy}$.

21)

show that each of these functions is nowhere analvtic:

(a) $f(z) = xy + iy$; (b) $f(z) = 2xy + i(x^2 - y^2)$; (c) $f(z) = e^y e^{ix}$.

23)

Show that $u(x, y)$ is harmonic in some domain and find a harmonic conjugate $v(x, y)$ when

(a) $u(x, y) = 2x(1 - y)$; (b) $u(x, y) = 2x - x^3 + 3xy^2$;

(c) $u(x, y) = \sinh x \sin y$; (d) $u(x, y) = y/(x^2 + y^2)$.

Ans. (a) $v(x, y) = x^2 - y^2 + 2y$; (b) $v(x, y) = 2y - 3x^2y + y^3$;

(c) $v(x, y) = -\cosh x \cos y$; (d) $v(x, y) = x/(x^2 + y^2)$.

25)

Write each of the following numbers in the form $a + bi$.

(a) $\exp(2 + i\pi/4)$ (b) $\dfrac{\exp(1 + i3\pi)}{\exp(-1 + i\pi/2)}$

(c) $\sin(2i)$ (d) $\cos(1 - i)$

(e) $\sinh(1 + \pi i)$ (f) $\cosh(i\pi/2)$

26)

Explain why the function $f(z) = \sin(z^2) + e^{-z} + iz$ is entire.

27)

Show the following.

(a) $\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$

(b) $\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$

28)

Evaluate each of the following.

(a) $\log i$ (b) $\log(1-i)$

(c) $\mathrm{Log}(-i)$ (d) $\mathrm{Log}(\sqrt{3}+i)$

30)

Find a branch of $\log(z^2+1)$ that is analytic at $z=0$ and takes the value $2\pi i$ there.

33)

Find the image of the half plane $y > 1$ under the transformation $w = (1-i)z$.

38)

Find a linear transformation that maps the circle $C_1 : |z-1| = 1$ onto the circle $C_2 : |w - 3i/2| = 2$.

39)

Find the image of the *interior* of the circle $C : |z-2| = 2$ under the Möbius transformation

$$w = f(z) = \frac{z}{2z-8}.$$