

# HW #5

## SOLUTIONS

### 1 FEM-HOMEWORK - 5

- 1.) The  $(x, y)$  coordinates of each-node are given for a four-node quadrilateral as shown. Obtain the isoparametric element in  $(\xi, \eta)$  coordinates.

#### Solution:

From the figure we could determine the coordinates of the boundary nodes.

$$x_1 = 1, y_1 = 1, x_2 = 5, y_2 = 1$$

$$x_3 = 6, y_3 = 6, x_4 = 1, y_4 = 4$$

The  $x$  could be written in terms of the nodal  $x$  values and the shape functions.

$$x(\xi, \eta) = N_1(\xi, \eta)x_1 + N_2(\xi, \eta)x_2 + N_3(\xi, \eta)x_3 + N_4(\xi, \eta)x_4$$

And the shape functions for a quadrilateral element is

$$N_1(\xi, \eta) = \frac{1}{4}(1 - \xi)(1 - \eta), N_1(-1, -1) = 0$$

$$N_2(\xi, \eta) = \frac{1}{4}(1 + \xi)(1 - \eta), N_2(1, -1) = 0$$

$$N_3(\xi, \eta) = \frac{1}{4}(1 + \xi)(1 + \eta), N_3(1, 1) = 0$$

$$N_4(\xi, \eta) = \frac{1}{4}(1 - \xi)(1 + \eta), N_4(-1, 1) = 0$$

The serendipity family of functions for boundary nodes

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} \frac{1}{4}(1 - \xi)(1 - \eta) & 0 & \frac{1}{4}(1 + \xi)(1 - \eta) & 0 \\ 0 & \frac{1}{4}(1 - \xi)(1 - \eta) & 0 & \frac{1}{4}(1 + \xi)(1 - \eta) \end{Bmatrix} \begin{Bmatrix} x_1 & y_1 & x_2 & y_2 & x_3 & y_3 & x_4 & y_4 \end{Bmatrix}$$

With the nodal values matrix becomes

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \frac{1}{4} \begin{Bmatrix} (1 - \xi)(1 - \eta) & 0 & (1 + \xi)(1 - \eta) & 0 \\ 0 & (1 - \xi)(1 - \eta) & 0 & (1 + \xi)(1 - \eta) \end{Bmatrix}$$

$$\begin{pmatrix} (1+\xi)(1+\eta) & 0 & (1-\xi)(1+\eta) & 0 \\ 0 & (1+\xi)(1+\eta) & 0 & (1-\xi)(1+\eta) \end{pmatrix} \left\{ \begin{array}{c} 1 \\ 1 \\ 5 \\ 1 \\ 6 \\ 6 \\ 1 \\ 4 \end{array} \right\}$$

We could get the x,y after the operation

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} \frac{1}{4}(13 + \eta + 9\xi + \xi\eta) \\ 3 + 2\eta + 0.5\xi + 0.5\xi\eta \end{Bmatrix}$$

3.) Calculate the value of

$$\int_{-3}^6 \int_2^5 \int_0^1 (x^2yz + yz^2 + 2xyz) dx dy dz$$

by using Gauss integration.

### Solution:

First of all in order to compare the results try an analytical solution.

#### *Analytical Solution*

$$\begin{aligned} \int_{-3}^6 \int_2^5 \int_0^1 (x^2yz + yz^2 + 2xyz) dx dy dz &= \int_{-3}^6 \int_2^5 (\frac{1}{3}x^3yz + yz^2x + yzx^2)|_0^1 dy dz \\ &= \int_{-3}^6 \int_2^5 (\frac{1}{3}yz + yz^2 + yz) dy dz \\ &= \int_{-3}^6 (\frac{1}{6}y^2z + \frac{1}{2}y^2z^2 + \frac{1}{2}y^2z)|_2^5 dx \\ &= \int_{-3}^6 (\frac{21}{6}z + \frac{21}{2}z^2 + \frac{21}{2}z) dz \\ &= \frac{21}{12}z^2 + \frac{21}{6}z^3 + \frac{21}{4}z^2|_{-3}^6 \\ &= \frac{21}{12}6^2 + \frac{21}{6}6^3 + \frac{21}{4}6^2 - \frac{21}{12}(-3)^2 - \frac{21}{6}(-3)^3 - \frac{21}{4}(-3)^2 \\ &= 63 + 756 + 189 - 63/4 + 189/2 - 189/4 \\ &= 1039.5 \end{aligned}$$

#### *Gauss Integration*

$$\int_{-3}^6 \int_2^5 \int_0^1 (x^2yz + yz^2 + 2xyz) dx dy dz$$

The coordinate transformation

$$x = \frac{1}{2}(1 + \xi) \text{ and } dx = \frac{1}{2}d\xi$$

$$y = \frac{1}{2}(7 + 3\eta) \text{ and } dy = \frac{3}{2}d\eta$$

$$z = \frac{3}{2}(1 + 3\zeta) \text{ and } dz = \frac{9}{2}d\zeta$$

Now write the integral with new coordinates

$$\int_{-3}^6 \int_2^5 \int_0^1 (x^2yz + yz^2 + 2xyz) dx dy dz = \frac{1}{2} \frac{3}{2} \frac{9}{2} \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 f(x(\xi), y(\eta), z(\zeta)) d\xi d\eta d\zeta$$

Thus the new function

$$f(\xi, \eta, \zeta) = \frac{27}{8} \{ [\frac{1}{4}(1 + \xi)]^2 \frac{1}{2}(7 + 3\eta) \frac{3}{2}(1 + 3\zeta)] + [\frac{1}{2}(7 + 3\eta) \frac{9}{4}(1 + 3\zeta)^2] + (1 + \xi) \frac{1}{2}(7 + 3\eta) \frac{1}{2}(7 + 3\eta) \}$$

and the integral is

$$\int_{-1}^1 \int_{-1}^1 \int_{-1}^1 f(\xi, \eta, \zeta) d\xi d\eta d\zeta$$

### *Applying the Gauss Integration*

Selecting one point approximation

The points:  $\pm 0.57750269$  and weight: 1

$$\begin{aligned} & \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 f(\xi, \eta, \zeta) d\xi d\eta d\zeta = f(0.57750269, 0.57750269, 0.57750269) \\ & + f(0.57750269, 0.57750269, -0.57750269) + f(0.57750269, -0.57750269, 0.57750269) \\ & + f(0.57750269, -0.57750269, -0.57750269) + f(-0.57750269, 0.57750269, 0.57750269) \\ & + f(-0.57750269, 0.57750269, -0.57750269) + f(-0.57750269, -0.57750269, 0.57750269) \\ & + f(-0.57750269, -0.57750269, -0.57750269) \end{aligned}$$

The result is

$$\int_{-1}^1 \int_{-1}^1 \int_{-1}^1 f(\xi, \eta, \zeta) d\xi d\eta d\zeta = 1039.5$$

2. The solution: **K stiffness Matrix**

$$\begin{array}{lll} x_1 := 3 & x_2 := 8 & x_3 := 4 \\ y_1 := 0 & y_2 := 3 & y_3 := 7 \\ & & y_4 := 2 \end{array}$$

$$N(\xi, \eta) := \begin{bmatrix} \frac{1}{4} \cdot (1-\xi) \cdot (1-\eta) & 0 & \frac{1}{4} \cdot (1+\xi) \cdot (1-\eta) & 0 & \frac{1}{4} \cdot (1+\xi) \cdot (1+\eta) & 0 & \frac{1}{4} \cdot (1-\xi) \cdot (1+\eta) & 0 \\ 0 & \frac{1}{4} \cdot (1-\xi) \cdot (1-\eta) & 0 & \frac{1}{4} \cdot (1+\xi) \cdot (1-\eta) & 0 & \frac{1}{4} \cdot (1+\xi) \cdot (1+\eta) & 0 & \frac{1}{4} \cdot (1-\xi) \cdot (1+\eta) \end{bmatrix} \quad xy := \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{pmatrix}$$

$$X(\xi, \eta) := N(\xi, \eta) \cdot xy$$

$$X(\xi, \eta) \text{ simplify } \rightarrow \begin{pmatrix} \frac{17}{4} - \frac{5}{4} \cdot \eta + \frac{7}{4} \cdot \xi - \frac{3}{4} \cdot \xi \cdot \eta \\ 3 + \frac{3}{2} \cdot \eta + 2 \cdot \xi + \frac{1}{2} \cdot \xi \cdot \eta \end{pmatrix}$$

$$x := \frac{17}{4} - \frac{5}{4} \cdot \eta + \frac{7}{4} \cdot \xi - \frac{3}{4} \cdot \xi \cdot \eta$$

$$y := 3 + \frac{3}{2} \cdot \eta + 2 \cdot \xi + \frac{1}{2} \cdot \xi \cdot \eta$$

$$J(\xi, \eta) := \begin{pmatrix} \frac{d}{d\xi} x & \frac{d}{d\xi} y \\ \frac{d}{d\eta} x & \frac{d}{d\eta} y \end{pmatrix} \quad J(\xi, \eta) \rightarrow \begin{pmatrix} \frac{7}{4} - \frac{3}{4} \cdot \eta & 2 + \frac{1}{2} \cdot \eta \\ -\frac{5}{4} - \frac{3}{4} \cdot \xi & \frac{3}{2} + \frac{1}{2} \cdot \xi \end{pmatrix} \quad |J(\xi, \eta)| \rightarrow \frac{41}{8} + \frac{19}{8} \cdot \xi - \frac{1}{2} \cdot \eta$$

In order to simplify the operations we assume the G constant and give E, v numerical values

$$E := 200 \quad v := 0.3 \quad G := \begin{pmatrix} \frac{E}{1-v^2} & \frac{E \cdot v}{1-v^2} & 0 \\ \frac{E \cdot v}{1-v^2} & \frac{E}{1-v^2} & 0 \\ 0 & 0 & \frac{E}{1+2 \cdot v} \end{pmatrix} \quad G = \begin{pmatrix} 219.78 & 65.934 & 0 \\ 65.934 & 219.78 & 0 \\ 0 & 0 & 125 \end{pmatrix}$$

Stiffness Matrix:

$$K = h \cdot \int_{-1}^1 \int_{-1}^1 N(\xi, \eta)^T \cdot D(\xi, \eta)^T \cdot G \cdot D(\xi, \eta) \cdot N(\xi, \eta) \cdot |J(\xi, \eta)| \, d\xi \, d\eta$$

$$D(\xi, \eta) := \begin{bmatrix} \frac{d}{d\eta} \left( 2 + \frac{1}{2} \cdot \eta \right)^{-1} & 0 \\ 0 & \frac{d}{d\xi} \left( \frac{-5}{4} - \frac{3}{4} \cdot \xi \right)^{-1} \\ \frac{d}{d\xi} \left( \frac{-5}{4} - \frac{3}{4} \cdot \xi \right)^{-1} & \frac{d}{d\eta} \left( 2 + \frac{1}{2} \cdot \eta \right)^{-1} \end{bmatrix}$$

$$DN(\xi, \eta) := \begin{bmatrix} \frac{-1}{2} \cdot (-1 + \xi) \cdot \frac{(-1 + \eta)}{(4 + \eta)^2} & 0 & \frac{1}{2} \cdot (1 + \xi) \cdot \frac{(-1 + \eta)}{(4 + \eta)^2} & 0 & \frac{-1}{2} \cdot (1 + \xi) \cdot \frac{(1 + \eta)}{(4 + \eta)^2} & 0 & \frac{1}{2} \cdot (-1 + \xi) \cdot \frac{(1 + \eta)}{(4 + \eta)^2} \\ 0 & 3 \cdot (-1 + \xi) \cdot \frac{(-1 + \eta)}{(5 + 3 \cdot \xi)^2} & 0 & -3 \cdot (1 + \xi) \cdot \frac{(-1 + \eta)}{(5 + 3 \cdot \xi)^2} & 0 & 3 \cdot (1 + \xi) \cdot \frac{(1 + \eta)}{(5 + 3 \cdot \xi)^2} & 0 & -3 \cdot (-1 + \xi) \cdot \frac{(1 + \eta)}{(5 + 3 \cdot \xi)^2} \\ 3 \cdot (-1 + \xi) \cdot \frac{(-1 + \eta)}{(5 + 3 \cdot \xi)^2} & \frac{-1}{2} \cdot (-1 + \xi) \cdot \frac{(-1 + \eta)}{(4 + \eta)^2} & -3 \cdot (1 + \xi) \cdot \frac{(-1 + \eta)}{(5 + 3 \cdot \xi)^2} & \frac{1}{2} \cdot (1 + \xi) \cdot \frac{(-1 + \eta)}{(4 + \eta)^2} & 3 \cdot (1 + \xi) \cdot \frac{(1 + \eta)}{(5 + 3 \cdot \xi)^2} & \frac{-1}{2} \cdot (1 + \xi) \cdot \frac{(1 + \eta)}{(4 + \eta)^2} & -3 \cdot (-1 + \xi) \cdot \frac{(1 + \eta)}{(5 + 3 \cdot \xi)^2} & \frac{1}{2} \cdot (-1 + \xi) \cdot \frac{(1 + \eta)}{(4 + \eta)^2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{-1}{2} \cdot (-1 + \xi) \cdot \frac{(-1 + \eta)}{(4 + \eta)^2} & 0 & 3 \cdot (-1 + \xi) \cdot \frac{(-1 + \eta)}{(5 + 3 \cdot \xi)^2} \\ 0 & 3 \cdot (-1 + \xi) \cdot \frac{(-1 + \eta)}{(5 + 3 \cdot \xi)^2} & \frac{-1}{2} \cdot (-1 + \xi) \cdot \frac{(-1 + \eta)}{(4 + \eta)^2} \\ \frac{1}{2} \cdot (1 + \xi) \cdot \frac{(-1 + \eta)}{(4 + \eta)^2} & 0 & -3 \cdot (1 + \xi) \cdot \frac{(-1 + \eta)}{(5 + 3 \cdot \xi)^2} \end{bmatrix}$$

$N(\xi, \eta)^T \cdot D(\xi, \eta)^T$  simplify  $\rightarrow$

$$\begin{bmatrix} \frac{-1}{2} \cdot (1 + \xi) \cdot \frac{(1 + \eta)}{(4 + \eta)^2} & -3 \cdot (1 + \xi) \cdot \frac{(-1 + \eta)}{(5 + 3 \cdot \xi)^2} & \frac{1}{2} \cdot (1 + \xi) \cdot \frac{(-1 + \eta)}{(4 + \eta)^2} \\ 0 & 3 \cdot (1 + \xi) \cdot \frac{(1 + \eta)}{(5 + 3 \cdot \xi)^2} & 3 \cdot (1 + \xi) \cdot \frac{(1 + \eta)}{(5 + 3 \cdot \xi)^2} \\ 0 & 3 \cdot (1 + \xi) \cdot \frac{(1 + \eta)}{(5 + 3 \cdot \xi)^2} & \frac{-1}{2} \cdot (1 + \xi) \cdot \frac{(1 + \eta)}{(4 + \eta)^2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} \cdot (-1 + \xi) \cdot \frac{(1 + \eta)}{(4 + \eta)^2} & 0 & -3 \cdot (-1 + \xi) \cdot \frac{(1 + \eta)}{(5 + 3 \cdot \xi)^2} \\ 0 & -3 \cdot (-1 + \xi) \cdot \frac{(1 + \eta)}{(5 + 3 \cdot \xi)^2} & \frac{1}{2} \cdot (-1 + \xi) \cdot \frac{(1 + \eta)}{(4 + \eta)^2} \end{bmatrix}$$

$$K = h \cdot \int_{-1}^1 \int_{-1}^1 f(\xi, \eta) d\xi d\eta \Rightarrow f(\xi, \eta) = (N(\xi, \eta))^T \cdot (D(\xi, \eta))^T \cdot G \cdot D(\xi, \eta) \cdot N(\xi, \eta) \cdot |J(\xi, \eta)|$$

$$K := f(0.577, 0.577) + f(-0.577, 0.577) + f(0.577, -0.577) + f(-0.577, -0.577), K = \begin{bmatrix} 274.733 & -60.859 & 84.059 & -22.167 & 38.553 & -8.55 & 127.394 & -23.359 \\ -60.859 & 469.446 & -22.167 & 139.073 & -8.55 & 64.851 & -23.359 & 219.444 \\ 84.059 & -22.167 & 61.198 & -27.727 & 26.679 & -10.812 & 38.553 & -8.55 \\ -22.167 & 139.073 & -27.727 & 86.342 & -10.812 & 39.726 & -8.55 & 64.851 \\ 38.553 & -8.55 & 26.679 & -10.812 & 45.423 & -15.482 & 70.014 & -12.004 \\ -8.55 & 64.851 & -10.812 & 39.726 & -15.482 & 72.419 & -12.004 & 120.096 \\ 127.394 & -23.359 & 38.553 & -8.55 & 70.014 & -12.004 & 234.379 & -32.491 \\ -23.359 & 219.444 & -8.55 & 64.851 & -12.004 & 120.096 & -32.491 & 407.53 \end{bmatrix}$$