PROJECT # 3

The two dimensional Euler equation is given by

$$\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0 \tag{1}$$

where

$$Q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix} \qquad F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ (E+p)u \end{bmatrix} \qquad G = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ (E+p)v \end{bmatrix} \tag{2}$$

In here, ρ is the density, (u, v) is the velocity vector components, E is the total energy per unit volume

$$E = \rho e + \frac{1}{2}\rho(u^2 + v^2) \tag{3}$$

and e is the internal energy per unit mass. The pressure term p is given by

$$p = (\gamma - 1)\left[E - \frac{1}{2}\rho(u^2 + v^2)\right] \tag{4}$$

where $\gamma = 1.4$.

Consider the internal flow in a channel with a circular bump given in the figure below: The problem consists of a channel of height L and length $3L$ with, along the bottom wall, a circular arc of length L and thickness equal to $L/10$. The free stream pressure and density values are 1 and 1.4, repectively. Solve the aforementioned problem for the following cases:

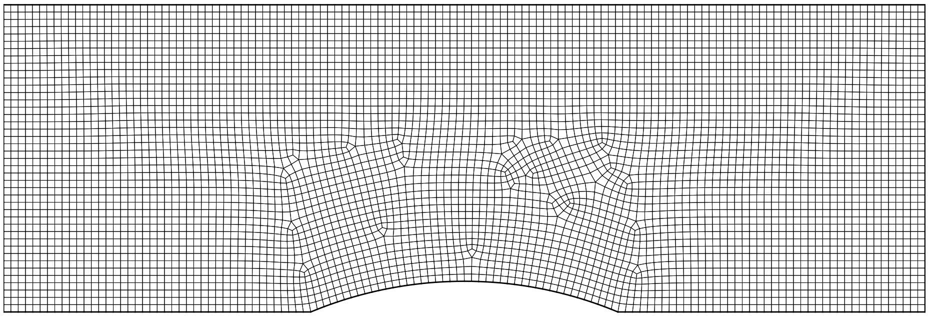

Figure 1: Unstructured computational mesh for the channel flow.

1. Subsonic case, $M = 0.1$
2. Transonic case, $M = 0.7$
3. Supersonic case, $M = 2.0$

For the present unstructured finite volume solver, store your state variables (Q) at the element centroids and use Roe's method to evaluate fluxes on the boundaries of the control volume.

The mesh file is available at: `http://www2.itu.edu.tr/~msahin/uut514e/bump.neu`
The grid subroutine to construct the mesh connectivity information for an unstructured mesh is available at: `http://www2.itu.edu.tr/~msahin/uut514e/grid.F`

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[0] UUT514E, 15 May 2010

Roe Approximate Riemann Solver:
The flux vector is given by

$$\mathbf{H}(Q) = F(Q)\mathbf{i} + G(Q)\mathbf{j} \tag{5}$$

The flux is given

$$h = \mathbf{n} \cdot \mathbf{H}(Q) \tag{6}$$

The inviscid fluxes on the boundaries of the control volume are given by

$$h(Q_L, Q_R; \mathbf{n}) = \frac{1}{2}\left[\mathbf{n} \cdot \mathbf{H}(Q_L) + \mathbf{n} \cdot \mathbf{H}(Q_R) - \hat{R}|\hat{\Lambda}|\hat{R}^{-1}(Q_R - Q_L)\right] \tag{7}$$

In here,

$$\hat{A} = \hat{R}\hat{\Lambda}\hat{R}^{-1} = \frac{\partial}{\partial Q}\left[\mathbf{n} \cdot \mathbf{H}(Q)\right] \tag{8}$$

The eigenvalues are given by

$$\hat{\Lambda} = \begin{bmatrix} \hat{q}_n - \hat{a} & 0 & 0 & 0 \\ 0 & \hat{q}_n & 0 & 0 \\ 0 & 0 & \hat{q} + \hat{a} & 0 \\ 0 & 0 & 0 & \hat{q}_n \end{bmatrix} \tag{9}$$

$$q_n = n_x u + n_y v \tag{10}$$

$$q_t = -n_y u + n_x v \tag{11}$$

The right eigenvectors are given by

$$\hat{R} = \left[\begin{array}{c|c|c|c} 1 & 1 & 1 & 0 \\ \hat{u} - \hat{a}n_x & \hat{u} & \hat{u} + \hat{a}n_x & -n_y \\ \hat{v} - \hat{a}n_y & \hat{v} & \hat{v} + \hat{a}n_y & n_x \\ \hat{H} - \hat{q}_n\hat{a} & 0.5(\hat{u}^2 + \hat{v}^2) & \hat{H} + \hat{q}_n\hat{a} & \hat{q}_t \end{array}\right] \tag{12}$$

The transformed variables $W = R^{-1}Q$ are

$$\Delta W_n = \hat{R}^{-1}\Delta Q = \begin{bmatrix} \dfrac{\Delta p - \hat{\rho}\hat{a}\Delta q_n}{2\hat{a}^2} \\ \Delta\rho - \dfrac{\Delta p}{\hat{a}^2} \\ \dfrac{\Delta p + \hat{\rho}\hat{a}\Delta q_n}{2\hat{a}^2} \\ \hat{\rho}\Delta q_t \end{bmatrix} \tag{13}$$

Roe averaged state variables are

$$\hat{\rho} = \sqrt{\rho_L \rho_R} \tag{14}$$

$$\hat{u} = \frac{\sqrt{\rho_L}u_L + \sqrt{\rho_R}u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \tag{15}$$

$$\hat{v} = \frac{\sqrt{\rho_L}v_L + \sqrt{\rho_R}v_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \tag{16}$$

$$\hat{H} = \frac{\sqrt{\rho_L}H_L + \sqrt{\rho_R}H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \tag{17}$$

$$\hat{a} = \sqrt{(\gamma - 1)\left[\hat{H} - 0.5(\hat{u}^2 + \hat{v}^2)\right]} \tag{18}$$

References

[1] P. L. Roe, Approximate Riemann Solvers, Parameter Vectors, and Difference Schemes. J. Comput. Phys., (1997), 135:250–258.

[2] H. Nishikawa and K. Kitamura, Very simple, carbuncle-free, boundary-layer-resolving, rotated-hybrid Riemann solvers. J. Comput. Phys., (2008), 227:2560–2581.