PROJECT # 1

The one dimensional heat equation is given by

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{1}$$

with the boundary conditions $T(0,t) = 0$ and $T(1,t) = T_0$ and the initial condition $T(x,0) = 0$.

1. Finite difference method on an uniform mesh

$$\frac{\partial T}{\partial t} = \frac{T_i^{n+1} - T_i^n}{\Delta t} \tag{2}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1}^{n+1} - 2T_i^{n+1} + T_{i-1}^{n+1}}{\Delta x^2} \tag{3}$$

Then the heat equation becomes

$$T_i^{n+1}\left[1 + 2\alpha\frac{\Delta t}{\Delta x^2}\right] + T_{i+1}^{n+1}\left[-\alpha\frac{\Delta t}{\Delta x^2}\right] + T_{i-1}^{n+1}\left[-\alpha\frac{\Delta t}{\Delta x^2}\right] = T_i^n \tag{4}$$

2. Finite volume method

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial T}{\partial t} dx = \int_{x_{i-1/2}}^{x_{i+1/2}} \alpha \frac{\partial^2 T}{\partial x^2} dx \tag{5}$$

$$\int_{x_{i-1/2}}^{x_i} \frac{\partial T}{\partial t} dx + \int_{x_i}^{x_{i+1/2}} \frac{\partial T}{\partial t} dx = \alpha \frac{\partial T}{\partial x}\Big|_{x_{i-1/2}}^{x_{i+1/2}} \tag{6}$$

The discretized equation becomes

$$\begin{aligned}\frac{1}{2}\left[\frac{\partial T_i}{\partial t} + \frac{1}{2}\left(\frac{\partial T_{i-1}}{\partial t} + \frac{\partial T_i}{\partial t}\right)\right]\frac{x_i - x_{i-1}}{2} + \frac{1}{2}\left[\frac{\partial T_i}{\partial t} + \frac{1}{2}\left(\frac{\partial T_i}{\partial t} + \frac{\partial T_{i+1}}{\partial t}\right)\right]\frac{x_{i+1} - x_i}{2} \\ = \alpha\frac{T_{i+1} - T_i}{x_{i+1} - x_i} - \alpha\frac{T_i - T_{i-1}}{x_i - x_{i-1}}\end{aligned} \tag{7}$$

After rearranging the above equation

$$\left[\frac{3}{4}\frac{\partial T_i}{\partial t} + \frac{1}{4}\frac{\partial T_{i-1}}{\partial t}\right]\frac{x_i - x_{i-1}}{2} + \left[\frac{3}{4}\frac{\partial T_i}{\partial t} + \frac{1}{4}\frac{\partial T_{i+1}}{\partial t}\right]\frac{x_{i+1} - x_i}{2} = \alpha\frac{T_{i+1} - T_i}{x_{i+1} - x_i} - \alpha\frac{T_i - T_{i-1}}{x_i - x_{i-1}} \tag{8}$$

3. Finite element method

$$\int_0^1 \frac{\partial T}{\partial t} N_i dx = \int_0^1 \alpha \frac{\partial^2 T}{\partial x^2} N_i dx \tag{9}$$

$$\int_0^1 \frac{\partial T}{\partial t} N_i dx = \int_0^1 \alpha \frac{\partial}{\partial x}\left[\frac{\partial T}{\partial x} N_i\right] dx - \int_0^1 \alpha \frac{\partial T}{\partial x}\frac{\partial N_i}{\partial x} dx \tag{10}$$

$$\int_0^1 \frac{\partial T}{\partial t} N_i dx = \alpha\left[\frac{\partial T}{\partial x} N_i\right]\Big|_0^1 - \int_0^1 \alpha \frac{\partial T}{\partial x}\frac{\partial N_i}{\partial x} dx \tag{11}$$

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Note that shape functions $N_i(0) = 0$ and $N_i(1) = 0$ for $i = 2, 3, 4, ..., N-1$, where N is the number of vertices

$$\int_0^1 \frac{\partial T}{\partial t} N_i dx = -\int_0^1 \alpha \frac{\partial T}{\partial x}\frac{\partial N_i}{\partial x} dx \tag{12}$$

For a jth finite element $N_j = (1-\xi)/2$ and $N_{j+1} = (1+\xi)/2$

$$T = T_j N_j + T_{j+1} N_{j+1} \tag{13}$$

$$x = x_j N_j + x_{j+1} N_{j+1} \tag{14}$$

And

$$dx = \left[x_j \frac{dN_j}{d\xi} + x_{j+1}\frac{dN_{j+1}}{d\xi}\right] d\xi = \frac{-x_j + x_{j+1}}{2} d\xi \tag{15}$$

The heat equation becomes

$$\sum_{j=1}^{N-1} \int_{-1}^{1} \left[\frac{\partial T_j}{\partial t} N_j + \frac{\partial T_{j+1}}{\partial t} N_{j+1}\right] N_i \frac{-x_j + x_{j+1}}{2} d\xi = -\sum_{j=1}^{N-1} \int_{-1}^{1} \alpha \left[T_j \frac{dN_j}{d\xi} + T_{j+1} \frac{dN_{j+1}}{d\xi}\right] \frac{d\xi}{dx} \frac{dN_i}{d\xi} d\xi \tag{16}$$

The above equation can be integrated using the Gauss-quadrature rule. Note that for the ith vertex equation ($i = 2, 3, 4, ..., N-1$) the above summation involves only the elements $j = i-1$ and $j = i$.