PROJECT # 3

The Euler equations governing unsteady compressible inviscid flows can be expressed in conservative form as:

$$\frac{\partial U}{\partial t} + \frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} = 0 \tag{1}$$

where

$$U = \begin{bmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ E \end{bmatrix} \quad E_1 = \begin{bmatrix} \rho u_1 \\ \rho u_1^2 + p \\ \rho u_1 u_2 \\ u_1(E+p) \end{bmatrix} \quad E_2 = \begin{bmatrix} \rho u_2 \\ \rho u_1 u_2 \\ \rho u_2^2 + p \\ u_2(E+p) \end{bmatrix} \tag{2}$$

and the internal energy of the gas

$$E = \rho e_{int} + \frac{1}{2}\rho(u_1^2 + u_2^2) \tag{3}$$

The Euler equations are not complete without an equation of state. We choose an ideal gas for which

$$e_{int} = \frac{p}{\rho(\gamma - 1)} \tag{4}$$

The initial conditions are $\rho = 1.4$, $u = 3$, $v = 0$ and $p = 1$. Use the fourth-order Runge-Kutta method with AUSM$^+$-up scheme [1], to solve the forward facing step problem given in Figure 1 at $t = 4$. The state variables U are defined at the cell centers.

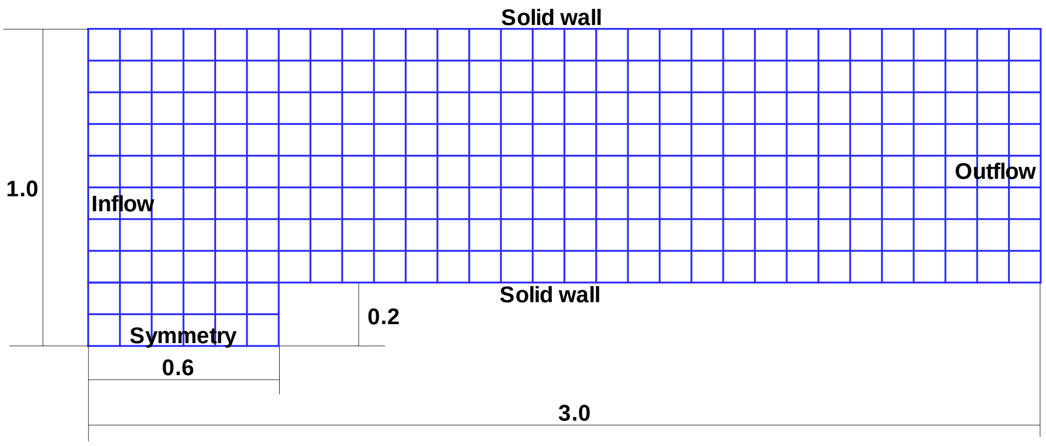

Figure 1: The forward step problem at $M_\infty = 3$ with boundary conditions.

The AUSM$^+$-up for all speeds

$$M_{L/R} = \frac{\mathbf{n} \cdot \mathbf{u}_{L/R}}{a_{1/2}} \tag{5}$$

where $a_{1/2} = (a_L + a_R)/2$

$$\bar{M}^2 = \frac{|\mathbf{u}_L|^2 + |\mathbf{u}_R|^2}{2a_{1/2}^2} \tag{6}$$

$$M_0^2 = min(1, max(\bar{M}^2, M_\infty^2)) \tag{7}$$

$$f_a(M_0) = M_0(2 - M_0) \tag{8}$$

---

[0] UTT619E, 4 January 2010

[1] Meng-Sing Liou, A sequel to AUSM, Part II: AUSM$^+$-up for all speeds. *J. Comput. Phys.*, **214**, (2006), 137–170.

$$\rho_{1/2} = \frac{\rho_L + \rho_R}{2} \tag{9}$$

$$M_{1/2} = M^{+}_{(4)}(M_L) + M^{-}_{(4)}(M_R) - \frac{K_p}{f_a} max(1 - \sigma \bar{M}^2, 0) \frac{p_R - p_L}{\rho_{1/2} a^2_{1/2}} \tag{10}$$

where

$$M^{\pm}_{(4)}(M) = \begin{cases} \frac{1}{2}(M \pm |M|) & \text{if} \quad |M| \geq 1 \\ \pm\frac{1}{4}(M \pm 1)^2 (1 + 16\beta\frac{1}{4}(M \mp 1)^2) & \text{otherwise} \end{cases} \tag{11}$$

and $\rho_{1/2} = (\rho_L + \rho_R)/2$, $K_p = 0.25$ and $\sigma = 1$. Then the mass flux

$$\dot{m}_{1/2} = \begin{cases} a_{1/2} M_{1/2} \rho_L & \text{if} \quad M_{1/2} > 0 \\ a_{1/2} M_{1/2} \rho_R & \text{otherwise} \end{cases} \tag{12}$$

the pressure flux

$$p_{1/2} = P^{+}_{(5)}(M_L) p_L + P^{-}_{(5)}(M_R) p_R - K_u P^{+}_{(5)}(M_L) P^{-}_{(5)}(M_R)(\rho_L + \rho_R)(f_a a^2_{1/2})(M_R - M_L) \tag{13}$$

and

$$P^{\pm}_{(5)}(M) = \begin{cases} \frac{1}{2M}(M \pm |M|) & \text{if} \quad |M| \geq 1 \\ \pm\frac{1}{4}(M \pm 1)^2 \left[(\pm 2 - M) + 16\alpha M \frac{1}{4}(M \mp 1)^2\right] & \text{otherwise} \end{cases} \tag{14}$$

using the parameters

$$\alpha = \frac{3}{16}(-4 + 5f_a^2) \qquad \beta = \frac{1}{8} \qquad K_u = 0.75 \tag{15}$$

The whole flux

$$\mathbf{n} \cdot \mathbf{E} = \dot{m}_{1/2} \begin{bmatrix} 1 \\ u_1 \\ u_2 \\ H \end{bmatrix}_{\substack{L \quad \text{if} \quad M_{1/2} > 0 \\ R \quad \text{otherwise}}} + \begin{bmatrix} 0 \\ p_{1/2} n_1 \\ p_{1/2} n_2 \\ 0 \end{bmatrix} \tag{16}$$

where the local enthalpy is given by

$$H = \frac{e + p}{\rho} \tag{17}$$