HOMEWORK # 5

1. Use seven iterations of the power method to compute the largest eigenvalue and the corresponding eigenvector of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \tag{1}$$

Use $x_0 = (1,1,1)^\top$ as the initial approximation.

The power method is described by the iteration

$$x_{n+1} = \frac{Ax_n}{||Ax_n||} \tag{2}$$

If x is an eigenvector, then its corresponding eigenvalue is given by

$$\lambda = \frac{Ax \cdot x}{x \cdot x} \tag{3}$$

This quotient is called the **Rayleigh quotient**.

```
INITIAL VECTOR
    X-VECTOR =    0.577350     0.577350     0.577350

ITERATION NUMBER =    1
    LAM =       3.000000
    X-VECTOR =    0.507093     0.169031     0.845154

ITERATION NUMBER =    2
    LAM =       2.142857
    X-VECTOR =    0.382360     0.382360     0.841191

ITERATION NUMBER =    3
    LAM =       2.929825
    X-VECTOR =    0.390567     0.442642     0.807171

ITERATION NUMBER =    4
    LAM =       3.101695
    X-VECTOR =    0.411040     0.411040     0.813691

ITERATION NUMBER =    5
    LAM =       3.006755
    X-VECTOR =    0.410104     0.404525     0.817419

ITERATION NUMBER =    6
    LAM =       2.988559
    X-VECTOR =    0.407937     0.407937     0.816808

ITERATION NUMBER =    7
    LAM =       2.999237
    X-VECTOR =    0.408041     0.408663     0.816393
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```
ITERATION NUMBER =    8
   LAM =          3.001270
   X–VECTOR =     0.408283      0.408283      0.816462

ITERATION NUMBER =    9
   LAM =          3.000085
   X–VECTOR =     0.408271      0.408202      0.816508

ITERATION NUMBER =   10
   LAM =          2.999859
   X–VECTOR =     0.408244      0.408244      0.816500

ITERATION NUMBER =   11
   LAM =          2.999991
   X–VECTOR =     0.408246      0.408253      0.816495

ITERATION NUMBER =   12
   LAM =          3.000016
   X–VECTOR =     0.408249      0.408249      0.816496

ITERATION NUMBER =   13
   LAM =          3.000001
   X–VECTOR =     0.408249      0.408248      0.816497

ITERATION NUMBER =   14
   LAM =          2.999998
   X–VECTOR =     0.408248      0.408248      0.816497

ITERATION NUMBER =   15
   LAM =          3.000000
   X–VECTOR =     0.408248      0.408248      0.816497

ITERATION NUMBER =   16
   LAM =          3.000000
   X–VECTOR =     0.408248      0.408248      0.816497

ITERATION NUMBER =   17
   LAM =          3.000000
   X–VECTOR =     0.408248      0.408248      0.816497
```

See ex51f.f code.

2. The Lorenz equations are the following system of differential equations

$$\frac{dx}{dt} = -10(x-y) \tag{4}$$

$$\frac{dy}{dt} = -xz + 28x - y \tag{5}$$

$$\frac{dz}{dt} = xy - 8z/3 \tag{6}$$

Use the fourth-order Runge-Kutta method with $x_0 = 0.0$, $y_0 = 20.0$, $z_0 = 25.0$ and $dt = 0.001$ to solve the above equations. Plot the solution in $x-y$ and $x-z$ plane.

The fourth-order Runge-Kutta method given by

$$x^{n+1} = x^n + \frac{h}{6}(k_{1,1} + 2k_{1,2} + 2k_{1,3} + k_{1,4}) \tag{7}$$

$$y^{n+1} = y^n + \frac{h}{6}(k_{2,1} + 2k_{2,2} + 2k_{2,3} + k_{2,4}) \tag{8}$$

$$z^{n+1} = z^n + \frac{h}{6}(k_{3,1} + 2k_{3,2} + 2k_{3,3} + k_{3,4}) \tag{9}$$

where

$$k_{1,1} = x(t, x^n, x^n, z^n) \tag{10}$$

$$k_{2,1} = y(t, x^n, x^n, z^n) \tag{11}$$

$$k_{3,1} = z(t, x^n, x^n, z^n) \tag{12}$$

$$k_{1,2} = x(t + \Delta t/2, x^n + k_{1,1}\Delta t/2, y^n + k_{2,1}\Delta t/2, z^n + k_{3,1}\Delta t/2) \tag{13}$$

$$k_{2,2} = y(t + \Delta t/2, x^n + k_{1,1}\Delta t/2, y^n + k_{2,1}\Delta t/2, z^n + k_{3,1}\Delta t/2) \tag{14}$$

$$k_{3,2} = z(t + \Delta t/2, x^n + k_{1,1}\Delta t/2, y^n + k_{2,1}\Delta t/2, z^n + k_{3,1}\Delta t/2) \tag{15}$$

$$k_{1,3} = x(t + \Delta t/2, x^n + k_{1,2}\Delta t/2, y^n + k_{2,2}\Delta t/2, z^n + k_{3,2}\Delta t/2) \tag{16}$$

$$k_{2,3} = y(t + \Delta t/2, x^n + k_{1,2}\Delta t/2, y^n + k_{2,2}\Delta t/2, z^n + k_{3,2}\Delta t/2) \tag{17}$$

$$k_{3,3} = z(t + \Delta t/2, x^n + k_{1,2}\Delta t/2, y^n + k_{2,2}\Delta t/2, z^n + k_{3,2}\Delta t/2) \tag{18}$$

$$k_{1,4} = x(t + \Delta t, x^n + k_{1,3}\Delta t, y^n + k_{2,3}\Delta t, z^n + k_{3,3}\Delta t) \tag{19}$$

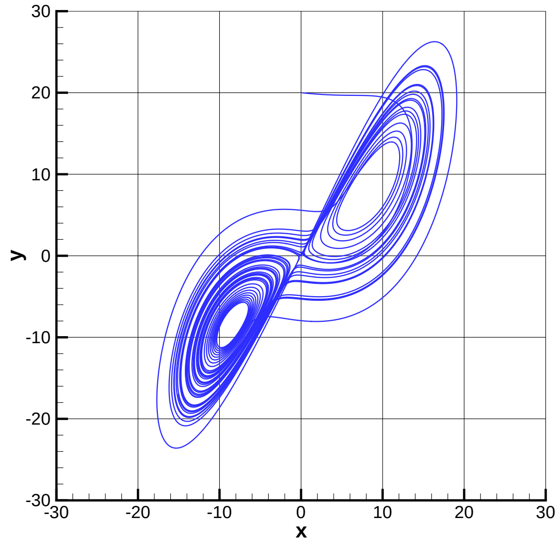
$$k_{2,4} = y(t + \Delta t, x^n + k_{1,3}\Delta t, y^n + k_{2,3}\Delta t, z^n + k_{3,3}\Delta t) \tag{20}$$

$$k_{3,4} = z(t + \Delta t, x^n + k_{1,3}\Delta t, y^n + k_{2,3}\Delta t, z^n + k_{3,3}\Delta t) \tag{21}$$

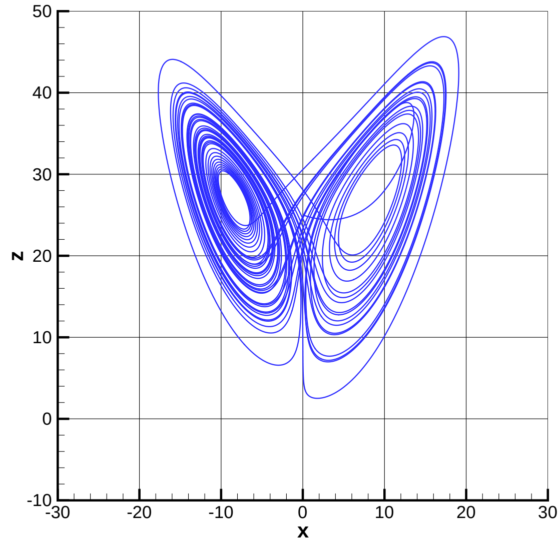
```
 ITERATION NUMBER=           1
X=     0.198905   Y=    19.980313   Z=     24.935412
 ITERATION NUMBER=           2
X=     0.395638   Y=    19.961264   Z=     24.874939
 ITERATION NUMBER=           3
X=     0.590227   Y=    19.942868   Z=     24.818520
 ITERATION NUMBER=           4
X=     0.782700   Y=    19.925138   Z=     24.766095
```

```
 ITERATION NUMBER=                5
X=      0.973085   Y=     19.908083   Z=     24.717606
 ITERATION NUMBER=                6
X=      1.161409   Y=     19.891712   Z=     24.672993
 ITERATION NUMBER=                7
X=      1.347700   Y=     19.876029   Z=     24.632202
 ITERATION NUMBER=                8
X=      1.531984   Y=     19.861038   Z=     24.595177
 ITERATION NUMBER=                9
X=      1.714290   Y=     19.846739   Z=     24.561864
 ITERATION NUMBER=               10
X=      1.894642   Y=     19.833132   Z=     24.532209
```

See ex52f.f code.



[a]



[b]

Figure 1: Solution of the Lorenz equations with the fourth-order Runge-Kutta method.

3. Solve the Laplace equation in a square enclosure with the boundary condition given in the figure below. The Laplace equation is given:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{22}$$

Assume that $\Delta x = \Delta y$.

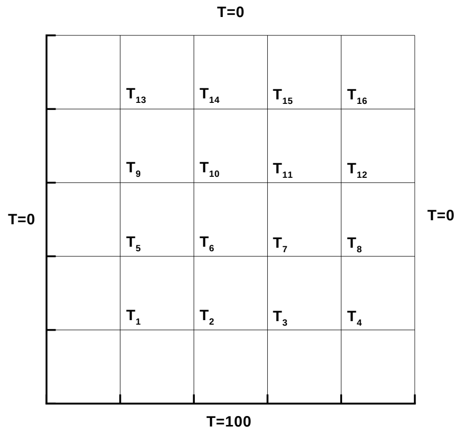


Figure 2: The computational domain with the Dirichlet boundary conditions.

The discrete equations resulting from the second-order central difference approximation

$$\begin{aligned}
-4T_1 + T_2 + T_5 &= -100 && (23)\\
T_1 - 4T_2 + T_3 + T_6 &= -100 && (24)\\
T_2 - 4T_3 + T_4 + T_7 &= -100 && (25)\\
T_3 - 4T_4 + T_8 &= -100 && (26)
\end{aligned}$$

$$\begin{aligned}
-4T_5 + T_6 + T_1 + T_9 &= 0 && (27)\\
T_5 - 4T_6 + T_7 + T_2 + T_{10} &= 0 && (28)\\
T_6 - 4T_7 + T_8 + T_3 + T_{11} &= 0 && (29)\\
T_7 - 4T_8 + T_4 + T_{12} &= 0 && (30)
\end{aligned}$$

$$\begin{aligned}
-4T_9 + T_{10} + T_5 + T_{13} &= 0 && (31)\\
T_9 - 4T_{10} + T_{11} + T_6 + T_{14} &= 0 && (32)\\
T_{10} - 4T_{11} + T_{12} + T_7 + T_{15} &= 0 && (33)\\
T_{11} - 4T_{12} + T_8 + T_{16} &= 0 && (34)
\end{aligned}$$

$$\begin{aligned}
-4T_{13} + T_{14} + T_9 &= 0 && (35)\\
T_{13} - 4T_{14} + T_{15} + T_{10} &= 0 && (36)\\
T_{14} - 4T_{15} + T_{16} + T_{11} &= 0 && (37)\\
T_{15} - 4T_{16} + T_{12} &= 0 && (38)
\end{aligned}$$

SOLUTION . . .

T 1	45.454545
T 2	59.469697
T 3	59.469697
T 4	45.454545
T 5	22.348485
T 6	32.954545
T 7	32.954545
T 8	22.348485
T 9	10.984848
T10	17.045455
T11	17.045455
T12	10.984848
T13	4.545455
T14	7.196970
T15	7.196970
T16	4.545455

See ex52f.f code.

4. Use the shooting method to solve the ordinary differential equation

$$y'' + 3xy' + 7y = cos(2x) \tag{39}$$

with $y(0) = 1$ and $y(\pi) = 0$.

First reduce the ordinary differential equation to the first-order

$$z' = -3xz - 7y + cos(2x) \tag{40}$$

$$y' = z \tag{41}$$

The initial values are set to $y(0) = 1$ and $z(0) = 0$. Then we will use the fourth-order Runge-Kutta method (given in question 2) to find the solution. However, $y(\pi) = 0$ condition will not be satisfied. Then, we will update the new value of $z(0)$ based on the computed value of $y(\pi)$ value. For this purpose, the modified Secant method can be used to find $z(0)$ value that will give $y(\pi) = 0$.

$$z(0)^{n+1} = z(0)^n - \frac{\delta F(z(0)^n)}{F(z(0)^n + \delta) - F(z(0)^n)} \tag{42}$$

where $F = y(\pi)$ computed from the fourth-order Runge-Kutta method.

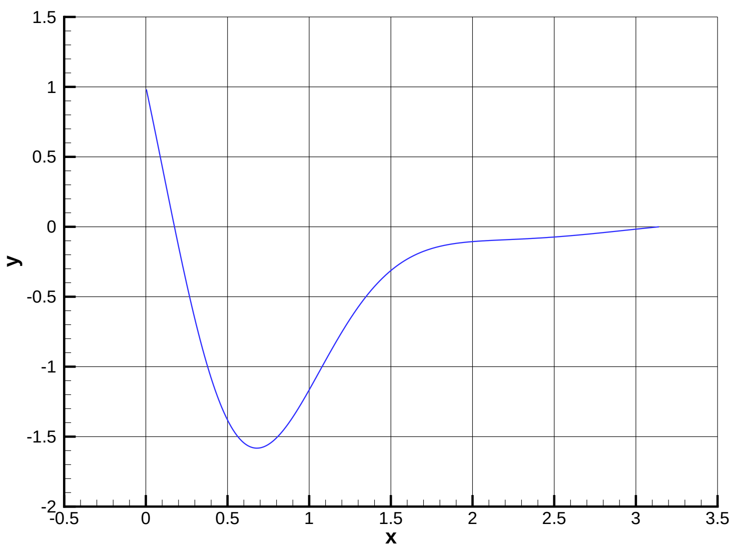


Figure 3: The solution obtained by the use of the shooting method.

```
X=   3.141593  Y=  -0.029416  Z=   0.140328
X=   3.141593  Y=  -0.029416  Z=   0.140328
 ROOT_Z0=  -5.4712212607372610
X=   3.141593  Y=  -0.000000  Z=   0.114973
X=   3.141593  Y=  -0.000000  Z=   0.114973
 ROOT_Z0=  -5.4712212674463832
X=   3.141593  Y=   0.000000  Z=   0.114973
X=   3.141593  Y=  -0.000000  Z=   0.114973
 ROOT_Z0=  -5.4712212674463538
```

(See ex54f.F code)