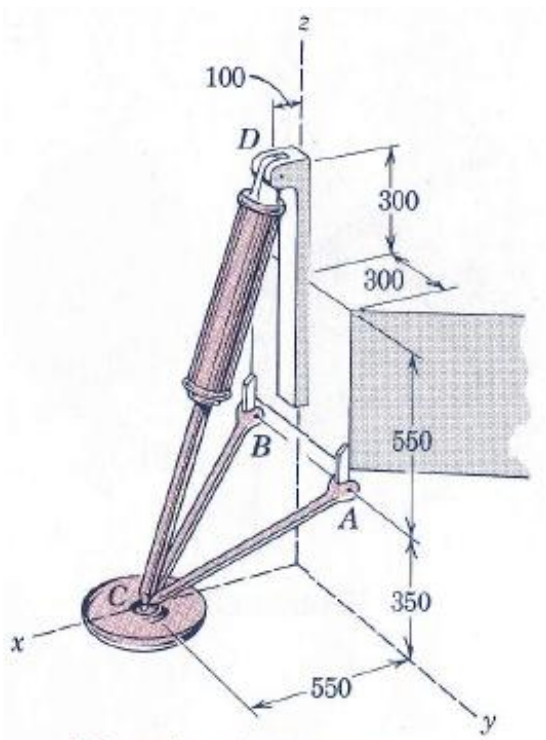
ITU Faculty of Aeronautics and Astronautics
2016-2017 Spring Semester
STA201E Statics

Homework #1 - SOLUTION

Problem: One of the three landing pads for the Mars Viking lander is shown in the figure with its approximate dimensions. The mass of the lander is 600 kg.



Dimensions in Millimeters

(a) Compute the force in each leg when the lander is resting on a horizontal surface on Mars. Assume equal support by the pads.

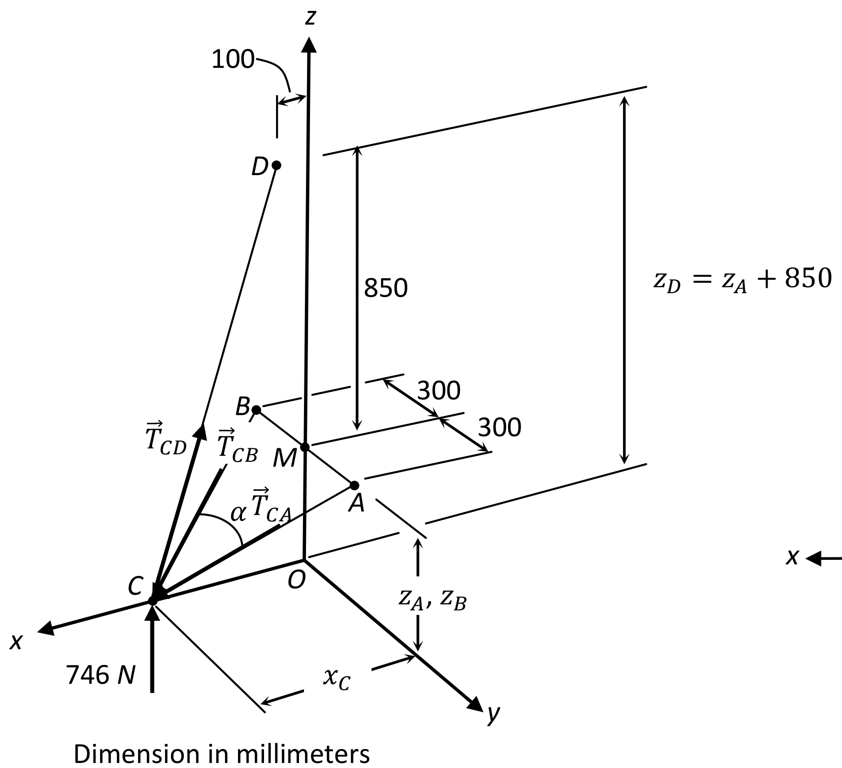
(b) The actuator is capable of retracting and extending the leg *CD* between the limits $l_{CD} = 1000\ mm$ and $l_{CD} = 1500\ mm$ when unconstrained. Considering the facts that the bottom of the lander is not to touch the surface and the leg *CD* cannot be positioned beyond the vertical position, find the minimum and maximum values of the *x*-coordinate of the point *C*. ($x_{C,min}$ and $x_{C,max}$)

(c) Determine the forces in each leg as a function of x_C.

(d) Draw the $F_{CD} - x_C$ and $F_{AB} - x_C$ diagrams for the interval $\left[x_{C,min},\ x_{C,max}\right\rangle$.

(e) What are the minimum and maximum values of the forces in the legs?

Solution:
(a)



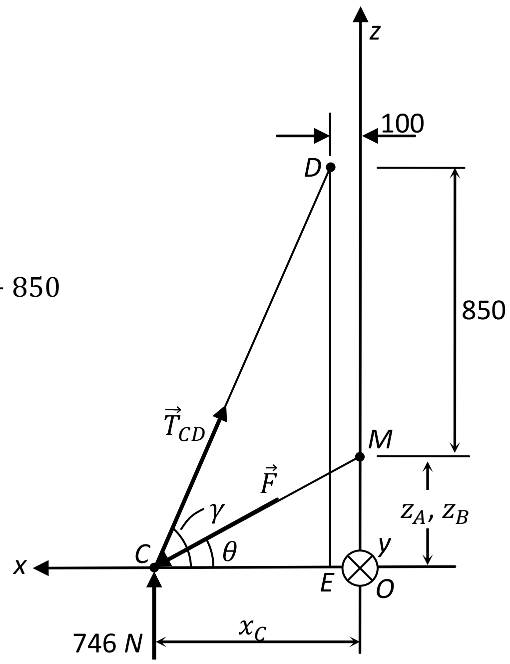


Figure 1a. Free Body Diagram (FBD) of particle *C*. **1b.** FBD of particle *C* in side view.

In figure 1a and 1b Free Body Diagram of particle *C* is given in three dimensions and in side view. Next step in the calculation of the forces in each arm is to obtain equation of equilbrium for particle *C*. Due to structural symmetry and symmetrical loading, forces in arms *AC* and *BC* will be equal independent of the position of the point *C*. Therefore,

$$|\vec{T}_{AC}| = |\vec{T}_{BC}| = T$$
$$|\vec{F}| = F = 2T\cos\left(\frac{\alpha}{2}\right)$$

The normal force that acts on each pad of the lander is found as follows:

Surface gravitational acceleration on Mars: $3.73\ m/s^2$

Normal force acting each pad of lander: $N = \frac{mg}{3} = \frac{(600)(3.73)}{3} = 746\ N$

To be able to determine the magnitudes and directions of the unknown forces in the arms from the equations of equilibrium the following geometrical quantities are calculated for $x_C = 550\ mm$ and $z_A = z_B = z_M = 350\ mm$:

Length of arms AC and BC: $L = \sqrt{(550)^2 + (300)^2 + (350)^2} = 717.6\ mm$

Distance CM: $|CM| = \sqrt{(550)^2 + (350)^2} = 651.9\ mm$

Angle BCA: $\angle BCA = \alpha = 2 \times \arctan\left(\frac{300}{651.9}\right) = 49.42°$

Angle MCO: $\angle MCO = \theta = \arctan\left(\frac{350}{550}\right) = 32.47°$

Angle DCE: $\angle DCE = \gamma = \arctan\left(\frac{1200}{450}\right) = 69.44°$

Length of arm CD: $|CD| = \sqrt{(450)^2 + (1200)^2} = 1281.6\ mm$

Now, the equations of equilbrium can be written:

$$\sum F_x = 0: 2T\cos\left(\frac{\alpha}{2}\right)\cos\theta - T_{CD}\cos\gamma = 0 \tag{1}$$

$$\sum F_z = 0: 746 - 2T\cos\left(\frac{\alpha}{2}\right)\sin\theta + T_{CD}\sin\gamma = 0 \tag{2}$$

Solving equations (1), (2)

$$T = -239.9\ N \quad \rightarrow \quad T = T_{AC} = T_{BC} = 240\ N \quad \text{(T)} \tag{3a}$$

$$T_{CD} = -1047.0 \quad \rightarrow \quad T_{CD} = 1047\ N\ \text{(C)} \tag{3b}$$

The minus signs indicate that the directions assumed for the legs are inappropriate; both forces must be in the opposite direction, namely, tensile forces in legs *AC* and *BC*, and compressive force in leg *CD*.

(b) When legs *AC* and *AB* are in the horizontal plane length of arm *CD* is equal to $|DH|$ (fig. 1c):

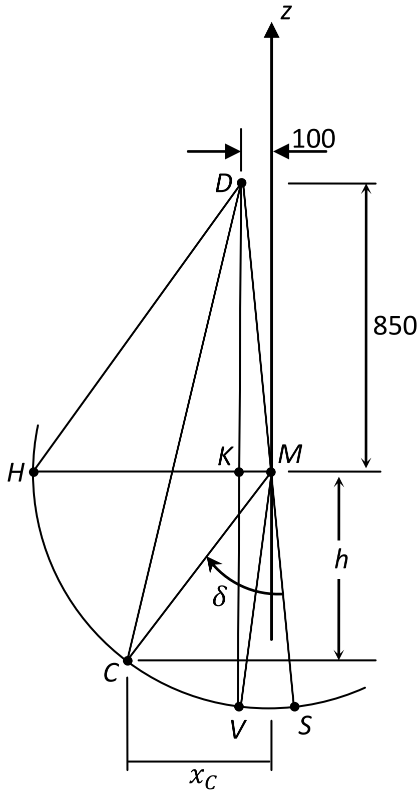
$$|DH| = \sqrt{|KH|^2 + |KD|^2} = \sqrt{(651.0 - 100)^2 + (850)^2} = 1013\ mm \tag{4a}$$

When the leg CD is in vertical position its length must be the following:

$$|DV| = |KD| + |KV| = 850 + \sqrt{|MV|^2 - |KM|^2} = 850 + \sqrt{(651.9)^2 - (100)^2} = 1494\ mm \tag{4b}$$

Both values are between the operational limits (1000-1500 mm), however, might it be possible that the length of leg *CD* exceeds these limit values when retracted/extended between positions *DH* and *DV*? The answer is no, and here is the proof:

When the point C is displaced it moves on a circular path with respect to the point M, as can be seen in fig. 1c. Length of leg CD related to the angular position of the line MC (projection of the legs AC and BC onto the xz-plane) can be obtained applying the cosine rule in the triangle MCD.



$$|CD|^2 = |MC|^2 + |MD|^2 + 2|MC||MD|\cos\delta$$

Since the side lengths $|MC|$ and $|MD|$ are constant values this expression can be rewritten in a simpler form:

$$|CD|^2 = C_1 + C_2\cos\delta$$

where C_1 adn C_2 are positive constants. Since $\cos\delta$ is monotonously decreasing function in the interval $\delta \in [0,180°]$ it is apparent that length of leg CD doesn't exceed its operational limits anywhere between position DV and DH. This completes the proof. (complete but what are the details?)

The trajectory of particle C is a circular arc, hence x_C will increase for increasing values of δ, and therefore, $x_{C,min} = 100\ mm$ and $x_{C,max} = L = 651.9\ mm$).

(c) The derivation of the forces in the legs as functions of the x-coordinate of point C is almost ready, the calculations that needs to be done is solving eqs. (1,2) for T_{AC}, T_{BC}, and T_{CD} and converting the trigonometric functions into x_C-terms. Thus, from eqs. (1) and (2):

$$T_{AC} = T_{BC} = T = \frac{746/2}{\cos(\alpha/2)\cos\theta\tan\gamma - \cos(\alpha/2)\sin\theta} \tag{5a}$$

$$T_{CD} = \frac{746\cos\theta}{\cos\theta\sin\gamma - \sin\theta\cos\gamma} \tag{5b}$$

By inspection of figs (1a) and (1b) the trigonometric terms are obtained as:

$$\cos(\alpha/2) = \frac{|MC|}{L} = \frac{651.9}{717.6} = 0.9084\ \left(= \frac{\sqrt{(550)^2+(350)^2}}{\sqrt{(550)^2+(300)^2+(350)^2}} = \sqrt{\frac{425000}{515000}} = \sqrt{\frac{85}{103}} = C_\alpha = constant\right) \tag{6a}$$

$$\cos\theta = x_C/|MC| \tag{6b}$$
$$\sin\theta = z_A/|MC| \tag{6c}$$
$$\cos\gamma = |CE|/|CD| \tag{6d}$$
$$\sin\gamma = |DE|/|CD| \tag{6e}$$
$$\tan\gamma = |DE|/|CE| \tag{6f}$$

Combining eqs. (5a-5b) and (6a-6f)

$$T_{AC} = T_{BC} = T = \frac{|CE||MC|\times 373/C_\alpha}{x_C|DE| - z_A|CE|} \tag{7a}$$

$$T_{CD} = \frac{746|CD|x_C}{x_C|DE| - z_A|CE|} \tag{7b}$$

is found. The trigonometric terms have been replaced by side lengths and coordinates but eqs. (7a) and (7b) are still not the final expressions, i.e. functions of one independent variable, x_C. Doing some further mathwork all sidelengths and coordinate z_A can be rewritten in terms of x_C:

$$z_A = \sqrt{|MC|^2 - x_C^2} = \sqrt{425000 - x_C^2} \tag{8a}$$

$$|CE| = x_C - 100 \tag{8b}$$

$$|DE| = 850 + z_A = 850 + \sqrt{425000 - x_C^2} \tag{8c}$$

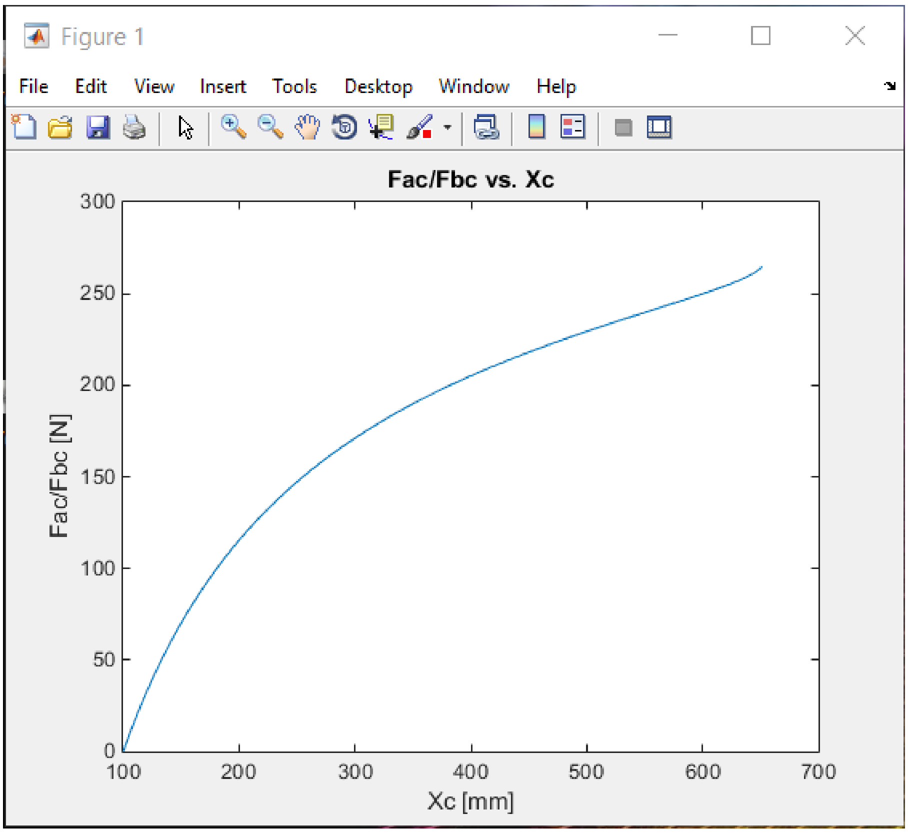
$$|CD| = \sqrt{|CE|^2 + |DE|^2} = \sqrt{(x_C - 100)^2 + \left(850 + \sqrt{425000 - x_C^2}\right)^2} \tag{8d}$$

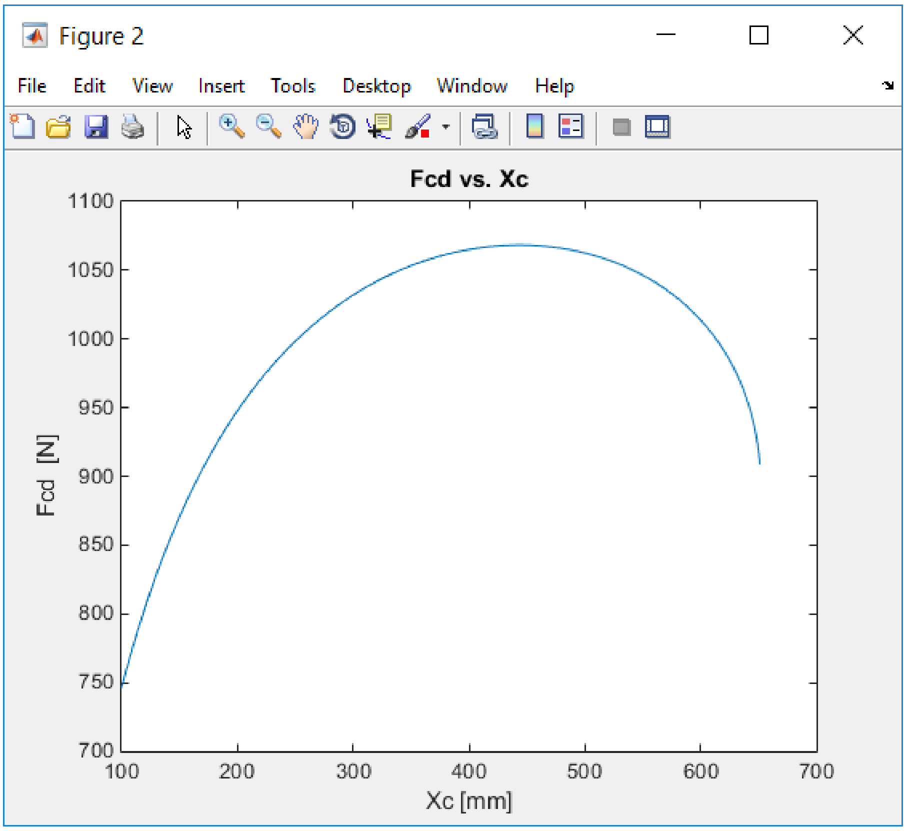
Substitution for the dependent variables in eqs. (7) to finish the mathematical manipulations:

$$T_{AC} = T_{BC} = T = \frac{(x_C-100)\sqrt{425000}\times 373\times\sqrt{\frac{103}{85}}}{x_C\left(850+\sqrt{425000-x_C^2}\right)-\left(\sqrt{425000-x_C^2}\right)(x_C-100)}$$

$$T_{CD} = \frac{746x_C\sqrt{(x_C-100)^2+\left(850+\sqrt{425000-x_C^2}\right)^2}}{x_C\left(850+\sqrt{425000-x_C^2}\right)-\left(\sqrt{425000-x_C^2}\right)(x_C-100)}$$

(d) Graphical representation can be obtained either by using a calculator and drawing the graph manually or by using an appropriate software. In this solution MATLAB R2014 was used. The script that was written and graphical results are given below:





MATLAB script that generates these figures:

```
xc= 100:0.1:651.9;
L = (350^2+550^2)^0.5; W=3.73*200; ca = (85/103)^0.5;

Fcd = W*(((xc-100).^2+(850+(L^2-xc.^2).^0.5).^2).^0.5)./(850+(L^2-xc.^2).^0.5-(1-100./xc).*(L^2-xc.^2).^0.5);
T =0.5*W*L*(xc-100)./(xc.*(850+(L^2-xc.^2).^0.5)-(xc-100).* (L^2-xc.^2).^0.5 )/ca;
figure (1)
plot(xc, T)
title('Fac/Fbc vs. Xc')
xlabel('Xc [mm]')
ylabel('Fac/Fbc [N]')
figure(2)
plot(xc, Fcd)
title('Fcd vs. Xc')
xlabel('Xc [mm]')
ylabel('Fcd  [N]')
[s,t]=max(Fcd);
[u,v]=max(T);
s
xc(t)
u
xc(v)
```

(e) The maximum values of the force in the legs are also found using MATLAB R2014:

$T_{AC,max} = T_{BC,max} = 266.4\ N$ at $x_C = 651.9\ mm$

$T_{CD,max} = 1068.3\ N$ at $x_C = 443.4\ mm$

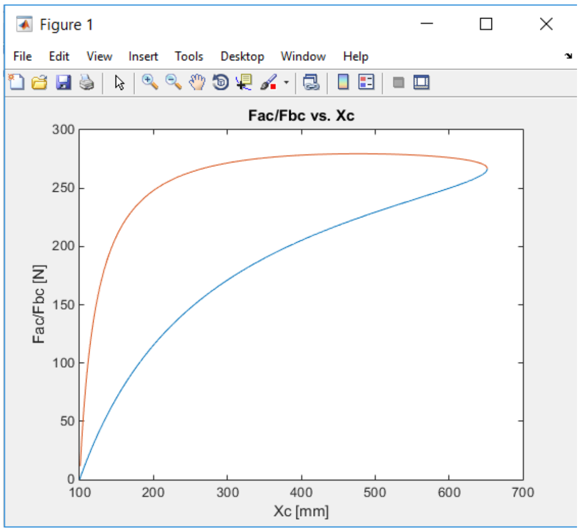
Addendum:

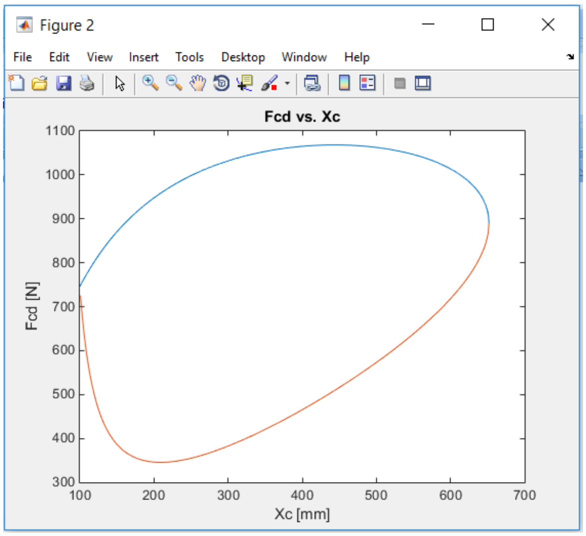
In the present homework assignment, the leg with the actuator was constrained to operate between the limits 1000-1500 mm. When the legs *AC* and *BC* were positioned horizontally the length of the leg CD had to be 1013 mm. The position at which leg CD is shortened to its minimum allowable length will occur slightly further than the horizonal alignment of legs *AC* and *BC*. But this is not permitted since the lander must not touch the ground. What if it were possible to rotate legs AC and *BC* to angles greather than the angle where *CD* is 1013 mm; how would the forces change in that case? (Assume that the pad is not directly connected to *C* but that there is a member of finite length between the pad and the point *C* preventing the lander from touching the surface. Ignore stability problems.) The analysis is similar, however the details are not given, instead the MATLAB script that includes analysis of forces corresponding to positions beyond operation conditions and the belonging result are presented.

MATLAB script including force calculations and graphical representations for positions that are beyond allowable operational conditions:

```
L = (350^2+550^2)^0.5; W=3.73*200; beta =atan(300/L);
xc= 100:0.01:L;
d = (L^2-xc.^2).^0.5;
alpha = atan(d./xc);
tetha = atan((850+d)./(xc-100));
Fcd = W*sin(pi/2+alpha)./sin(tetha-alpha);
T = W*sin(pi/2-tetha)./sin(tetha-alpha);
Fac = T./(2*cos(beta));
diff = Fcd.*sin(tetha)-T.*sin(alpha);

xyc = L:-1:100
tet = acos(xyc/L);
alf = atan((850-L*sin(tet))./(L*cos(tet)-100));
Tucd = W*sin(pi/2-tet)./sin(alf+tet);
Fuac =W*sin(pi/2-alf)./sin(alf+tet)/(2*cos(beta));

figure (1)
plot(xc, Fac)
title('Fac/Fbc vs. Xc')
xlabel('Xc [mm]')
ylabel('Fac/Fbc [N]')
hold
plot(xyc, Fuac)

figure (2)
plot(xc, Fcd)
title('Fcd vs. Xc')
xlabel('Xc [mm]')
ylabel('Fcd [N]')
hold
plot(xyc, Tucd)
```

PLOT results for legs *AC-BC* and *CD*:





The blue curves in these two figures correspond to the interval $x_C \in [100,\ L]$. In contrast, the orange curves are plotted for the imaginary case, i.e.when the assembly is further displaced to a position that exceeds safety limits. In this region, *AC/BC* curve continues to increase, although not very much perceivable, then a moderate decrement is observed and finally a sharp drop occurs about $x_C = 150\ mm$ until the terminal value $F_{AC/BC} = 0$ at $x_C = 100\ mm$. In the figure on the right, force in leg *CD* decreases to a minimum value about $x_C = 200\ mm$, approximately, after which a steep increase can be seen until the position $x_C = 100\ mm$, where the magnitude $F_{DC} = 746\ N$ is recovered. This is expected because once more the leg *CD* is vertical and whole weight rests on leg *CD* only. The forces in the legs *AC* and *BC* must be zero. (can this be proven analytically?)

This addemdum part encompasses the mathematical solution of the original problem in the "imaginary" region, a configuration very impractical to construct. A more complete analysis would be one in which the true structure of the landing mechanism is given, with all other constraints that are not considered here.
Of course necessity of software very useful to carry out such detailed structural analysis needs not to be mentioned.

================================ END OF SOLUTION ===================================