Istanbul Technical University Faculty of Aeronautics and Astronautics **04.12.2017**
2017-2018 Fall Semester
11729 DNK201E Dynamics

Midterm Exam – SOLUTION

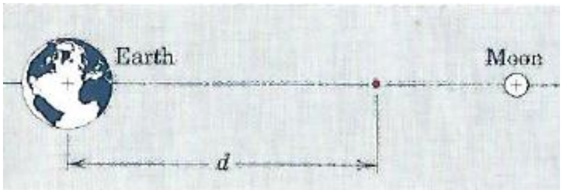
Problem 1: **(10 p)** The position s of particle along a straight line is given by $s = 8e^{-0.4t} - 6t + t^2$, where s is in meters and t is the time in seconds. Determine the position s and velocity v when the acceleration is $3\ m/s^2$.

Solution 1:

$$v = \frac{ds}{dt} = -3.2e^{-0.4t} - 6 + 2t, \quad a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 1.28e^{-0.4t} + 2$$

$$a = 3 \quad \rightarrow \quad t = \left[\ln\left\{\frac{3-2}{1.28}\right\}\right]\left(\frac{1}{-0.4}\right) = 0.617\ s \quad \rightarrow \quad v(0.617\ s) = -7.27\frac{m}{s}, \quad s(0.617\ s) = \ 2.93\ m$$

Problem 2: **(10 p)** Calculate the two distances *d* from the center of the earth at which a particle experiences equal attractions from the earth and moon. The particle is restricted to the line that joins the centers of the earth and moon.



Solution 2: Distance between moon and earth center to center: $x = 384398\ km$

$$F_{earth} = F_{moon} \quad \rightarrow G\frac{m_{earth}m_{particle}}{d^2} = G\frac{m_{moon}m_{particle}}{(x-d)^2} \quad \rightarrow \quad 0.0123d^2 = x^2 - 2xd + d^2$$
$$\rightarrow (1 - 0.0123)d^2 - 2xd + x^2 = 0 \qquad (1)$$

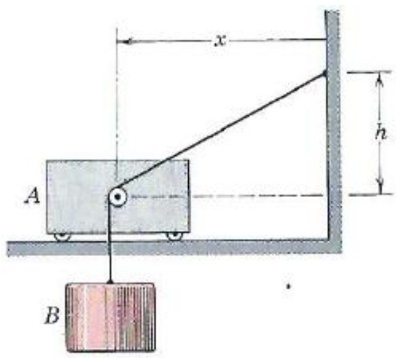
The solution of eq. (1) which is a polynomial of second degree is twofold:

$$d_{1,2} = \frac{2x \pm \sqrt{4x^2 - 4(1-0.0123)x^2}}{2(1-0.0123)} = \frac{x \pm \sqrt{(0.0123)x^2}}{(1-0.0123)} = \frac{x(1 \pm \sqrt{0.0123})}{(1-0.0123)} = \frac{x}{1 \pm \sqrt{0.0123}}$$

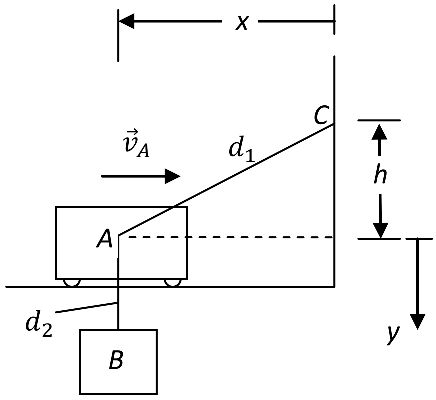
$$d_1 = \frac{384398}{1 \pm \sqrt{0.0123}} = 346022\ km, \quad d_2 = \frac{384398}{1 - \sqrt{0.0123}} = 432348km$$

The second position is to the right of the moon. This is possible since mass of the earth is greater than the mass of the moon, resulting in greater attraction at farther distance because the attraction of the moon dies out faster.

Problem 3: **(15 p)** Neglect the diameter of the small pulley attached to body *A* and determine the magnitude of the total velocity of *B* in terms of velocity v_A that body *A* has to the right. Assume that the cable between *B* and the pulley remains vertical and solve for a given value of x.



Solution 3: $d_1 = |AC| = \sqrt{x^2 + h^2}, d_2 = |AB| = y_B$



Length of cable is constant: $d_1 + d_2 = const \rightarrow \dot{d}_1 + \dot{d}_2 = 0$

Velocity of particle A: $\vec{v}_A = v_A(-\vec{\imath}) = v_{A,x}\vec{\imath} = \dot{x}\vec{\imath}$ (v_A is the magnitude)

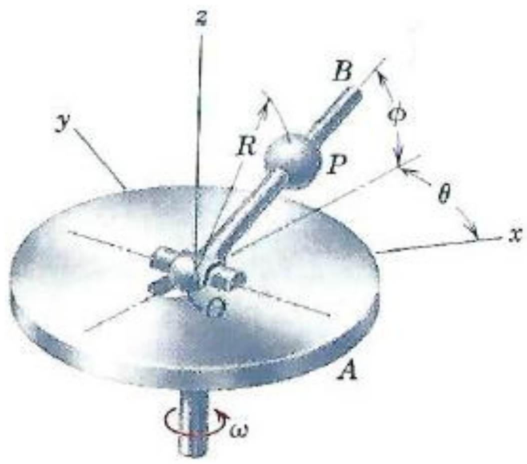
Velocity of particle B: $\vec{v}_B = v_{B,x}\vec{\imath} + v_{B,y}\vec{\jmath} = \dot{x}\vec{\imath} + \dot{y}\vec{\jmath}$

$\rightarrow v_{B,x} = v_{A,x} = -v_A$

$$v_{B,y} = \dot{d}_2 = -\dot{d}_1 = -\frac{1}{2}\frac{2x\dot{x}}{\sqrt{x^2+h^2}} = -\frac{x\dot{x}}{\sqrt{x^2+h^2}} = \frac{xv_A}{\sqrt{x^2+h^2}}$$

$$v_B = \sqrt{v_{B,x}^2 + v_{B,y}^2} = \sqrt{(-v_A)^2 + \left(\frac{xv_A}{\sqrt{x^2+h^2}}\right)^2} = v_A\sqrt{\frac{2x^2+h^2}{x^2+h^2}}$$

Problem 4: (30 p) The disk *A* rotates about the vertical *z*-axis with constant speed $\omega = \dot{\theta} = \pi/3\ rad/s$. Simultaneously, the hinged arm *OB* is elevated at the constant rate $\dot{\phi} = 2\pi/3\ rad/s$. At time $t = 0$, both $\theta = 0$ and $\phi = 0$. The angle θ is measured from the fixed reference *x*-axis. The small sphere *P* slides out along the rod according to $R = 50 + 200t^2$, where R is in millimeters and t is in seconds. When $t = \frac{1}{2}s$,



- **(a)** determine the total acceleration $\vec{a}$ of *P* ,
- **(b)** calculate the magnitude of the acceleration vector,
- **(c)** find the velocity vector $\vec{v}$ of *P*, and
- **(d)** find the angular momentum of the sphere in terms of the mass m of the sphere.

Solution 4:

(a) Angular velocities are constant. Therefore,

$\theta = \int_0^t \dot{\theta}dt = \dot{\theta}t$ and $\phi = \int_0^t \dot{\phi}dt = \dot{\phi}t \rightarrow\ \theta = \frac{\pi}{3}\frac{1}{2} = \frac{\pi}{6}\ rad\ ,\ \phi = \frac{2\pi}{3}\frac{1}{2} = \frac{\pi}{3}\ rad$

Radius at t=0.5 s: $R = 50 + 200(0.5)(0.5) = 100\ mm = 0.1\ m$
Time rate of change of radius at t=0.5 s: $\dot{R} = 400t = 400(0.5) = 200\frac{mm}{s} = 0.2\frac{m}{s}$
Time rate of change of $\dot{R}$at t = 0.5 s: $\ddot{R} = 400\frac{mm}{s^2} = 0.4\frac{m}{s^2}$

$$a_R = \ddot{R} - R\dot{\phi}^2 - R\dot{\theta}^2\cos^2\phi = 0.4 - (0.1)(2\pi/3)^2 - (0.1)(\pi/3)^2(\cos^2\pi/3) = -0.06606\ m/s^2$$

$$a_\phi = R\ddot{\phi} + 2\dot{R}\dot{\phi} + R\dot{\theta}^2\sin\phi\cos\phi =$$
$$= (0.1)(0) + 2(0.2)(2\pi/3) + (0.1)(\pi/3)^2\sin\pi/3\cos\pi/3 = 0.8852\ m/s^2$$

$$a_\theta = R\ddot{\theta}\cos\phi + 2\dot{R}\dot{\theta}\cos\phi - 2R\dot{\phi}\dot{\theta}\sin\phi$$
$$= (0.1)(0)(\cos\pi/3) + 2(0.2)(\pi/3)(\cos\pi/3) - 2(0.1)(\pi/3)(2\pi/3)\sin\pi/3 = -0.1704\ m/s^2$$

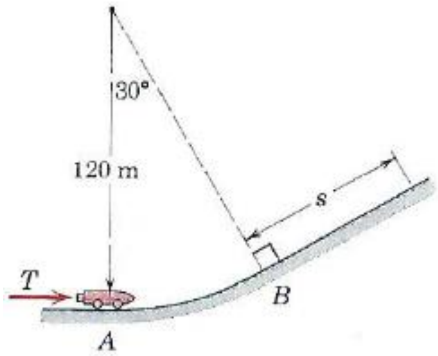
$$\vec{a} = -0.06606\vec{e}_R + 0.8852\ \vec{e}_\phi - 0.1704\vec{e}_\theta\ \ m/s^2$$

(b) $a = \sqrt{a_R^2 + a_\phi^2 + a_\theta^2} = \sqrt{(-0.06606)^2 + (0.8852)^2 + (-0.1704)^2} = 0.904\, m/s^2$

(c) $\vec{v} = \dot{R}\vec{e}_R + R\dot{\phi}\vec{e}_\phi + R\dot{\theta}\cos\phi\,\vec{e}_\theta = 0.2\vec{e}_R + (0.1)(2\pi/3)\vec{e}_\phi + (0.1)\left(\frac{\pi}{3}\right)\left(\cos\frac{\pi}{3}\right)\vec{e}_\theta$
$= 0.2\vec{e}_R + 0.209\vec{e}_\phi + 0.05235\vec{e}_\theta \quad m/s$

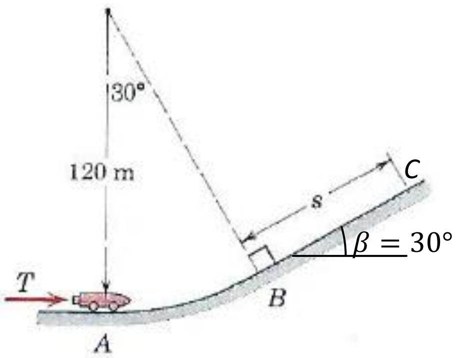
(d) $\vec{H}_0 = \vec{r} \times m\vec{v} = R\vec{e}_R \times m\left(\dot{R}\vec{e}_R + R\dot{\phi}\vec{e}_\phi + R\dot{\theta}\cos\phi\,\vec{e}_\theta\right) =$
$= \left(mR\dot{R}\right)(\vec{e}_R \times \vec{e}_R) + \left(mR^2\dot{\phi}\right)(\vec{e}_R \times \vec{e}_\phi) + \left(mR^2\dot{\theta}\cos\phi\right)(\vec{e}_R \times \vec{e}_\theta)$
$= m\left(-0.0209\vec{e}_\theta + 0.005235\vec{e}_\phi\right)\ kg \cdot m^2/s$

<u>Problem 5:</u> (20 p) A small rocket-propelled test vehicle with a total mass of $100\ kg$ starts from rest at A and moves with negligible friction along the track in the vertical plane as shown. If the propelling rocket exerts a constant thrust T of $1.5\ kN$ from position A to position B where it is shut off, determine the distance s that the vehicle rolls up the incline before stopping. The loss of mass due the expulsion of gases by the rocket is small and may be neglected.



<u>Solution 5:</u>

Work-Energy principle: $W_{AC} = T_C - T_A$ (a)
(C is the point where the rocket stops)



$T_C = T_A = 0$

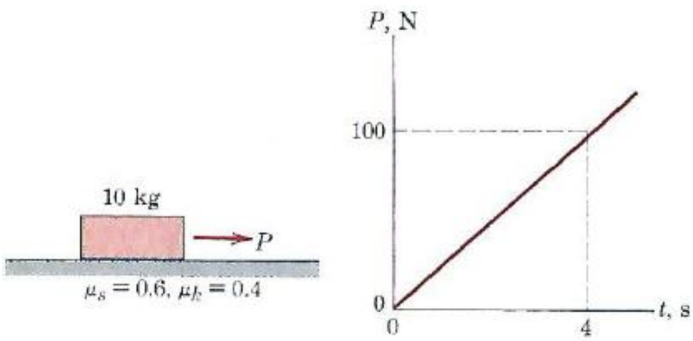
$W_{AC} = Ts_{AB} - W\Delta h_{AC} = T \cdot R\theta - mg(h_B + s\sin\theta) =$
$= T \cdot R\theta - mg(R - R\sin\theta + s\sin\beta)$ (b)

From (a) and (b) → $(1500)(120)\left(\frac{\pi}{6}\right) - (100)(9.81)\left(120 - 120\cos\frac{\pi}{6} + s\sin\frac{\pi}{6}\right) = 0 - 0 = 0$

$\rightarrow s = 159.9 \cong 160\ m$

<u>Problem 6:</u> (15 p) The force P, which is applied to the 10-kg block initially at rest, varies linearly with the time as indicated. If the coefficients of static and kinetic friction between the block and the horizontal surface are 0.6 and 0.4 respectively,

- **(a)** determine the velocity of the block when $t = 4\ s$, and
- **(b)** power delivered by the force P when $t = 3\ s$.



Solution 6: **(a)** Block will start moving to the right at the instant at which the force *P* will be equal to the maximum static friction:

$P(t) = \frac{100}{4}t = (0.6)(10)(9.81) \quad \rightarrow t = 2.3544\ s$ (motion impending at this instant)

After motion has started kinetic friction will be the resisting force resulting in the following acceleration:

$\sum F_x = ma_x: \quad P(t) - \mu_k N = ma_x$

$$a_x = \frac{2.5t - (0.4)(10)(9.81)}{10} = 2.5t - 3.924 \quad m/s^2$$

$\int_{v(t=2.3544)=0}^{v(t=4)} dv = \int_{t=2.3544}^{t=4} (2.5t - 3.924)\, dt \quad \rightarrow \quad v(t=4) = [1.25t^2 - 3.924t]_{2.3544}^{4} = 6.61\ m/s$ (i)

Power delivered to the block by means of the force *P* at the instant t=3 s:

$$power = \frac{dU}{dt} = \vec{P}(3\ s) \cdot \vec{v}(3s) = P(3s)\vec{\imath} \cdot v(3s)\vec{\imath} = P(3s)v(3s)$$

The magnitude of the force P at the instant t=3 s can be calculated easily: $P(3s) = 75\ N$

The velocity of the block at the instant t=3s can be found by setting t=3s as the upper limit in eq. (i):

$v(3s) = 1.787\ m/s \rightarrow power = (75)(1.787) = 134\ W$

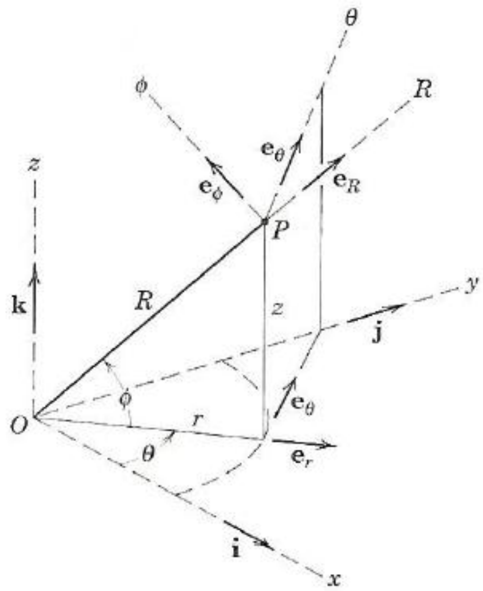
Solar System Constants:

Universal gravitational constant: $G = 6.673 \cdot 10^{-11}\ m^3/(kg \cdot s^2)$
Mass of Earth: $m_{earth} = 5.976 \cdot 10^{24}\ kg$
Mass of Moon: $m_{moon} = 0.0123 m_{earth}$
Mean distance from Moon to Earth (center-to-center): $384398\ km$

Kinematics and Kinetics of a Particle:

Coordinates	Position ($\vec{r}$)	Velocity ($\vec{v}$)	Acceleration ($\vec{a}$)
Rectangular	$x\vec{\imath} + y\vec{\jmath} + z\vec{k}$	$\dot{x}\vec{\imath} + \dot{y}\vec{\jmath} + \dot{z}\vec{k}$	$\ddot{x}\vec{\imath} + \ddot{y}\vec{\jmath} + \ddot{z}\vec{k}$
Polar	$r\vec{e}_r$	$\dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$	$(\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta$
Cylindrical	$r\vec{e}_r + z\vec{k}$	$\dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta + \dot{z}\vec{k}$	$(\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta + \ddot{z}\vec{k}$
Spherical	$R\vec{e}_R$	$\dot{R}\vec{e}_R + R\dot{\phi}\vec{e}_\phi$ $+ R\dot{\theta}\cos\phi\, \vec{e}_\theta$	$(\ddot{R} - R\dot{\phi}^2 - R\dot{\theta}^2\cos^2\phi)\vec{e}_R$ $+(R\ddot{\phi} + 2\dot{R}\dot{\phi} + R\dot{\theta}^2\sin\phi\cos\phi)\vec{e}_\phi$ $+(R\ddot{\theta}\cos\phi + 2\dot{R}\dot{\theta}\cos\phi - 2R\dot{\phi}\dot{\theta}\sin\phi)\vec{e}_\theta$

Tangential and normal components:
$\vec{v} = v\vec{n}_t$, $\vec{a} = \dot{v}\vec{n}_t + (v^2/\rho)\vec{n}_n$
Radius of curvature: $\rho = \frac{\left[1+(dy/dx)^2\right]^{3/2}}{d^2y/dx^2}$
Work-energy principle: $U_{12} = T_2 - T_1$
Work of a force: $U_{12} = \int_1^2 \vec{F} \cdot d\vec{r}$
Power: $\frac{dU}{dt} = \vec{F} \cdot \vec{v}$
Conservative force: $\vec{F} = -\vec{\nabla}\mathrm{V}$, Conservation of mechanical energy: $T_1 + V_1 = T_2 + V_2$
Time rate of change of linear momentum:
$\sum \vec{F} = \dot{\vec{G}} = \frac{d}{dt}(m\vec{v})$



Time rate of change of angular momentum:
$\dot{\vec{H}}_0 = \frac{d}{dt}(\vec{r} \times m\vec{v}) = \vec{r} \times \vec{F}, \quad \sum \vec{M}_0 = \dot{\vec{H}}_0$
$\sum M_x = \dot{H}_x \quad \sum M_y = \dot{H}_y \quad \sum M_z = \dot{H}_z$
Angular impulse: $\sum_{i=1}^{n} \int_{t_1}^{t_2} \vec{M}_{i_O} dt = \vec{H}_{O_2} - \vec{H}_{O_1} = \sum_{i=1}^{n} (\vec{r} \times m_i \vec{v}_i)_2 - \sum_{i=1}^{n} (\vec{r} \times m_i \vec{v}_i)_1$
Coefficient of restitution: $e = \frac{v_2' - v_1'}{v_1 - v_2}$