MAT 271E – Probability and Statistics

Spring 2012

Instructor :	İlker Bayram EEB 1103 `ibayram@itu.edu.tr`
Class Meets :	13.30 – 16.30, Wednesday EEB 5305
Office Hours :	14.00 – 16.00, Monday
Textbook :	D. B. Bertsekas, J. N. Tsitsiklis, 'Introduction to Probability', 2nd Edition, Athena-Scientific.
Supp. Text :	• D. Wackerly, W. Mendenhall, R. L. Scheaffer, 'Mathematical Statistics with Applications'. • S. Ross, 'Introduction to Probability and Statistics for Engineers and Scientists'.
Grading :	Homeworks (5%), 2 Midterms (27.5% each), Final (40%).
Webpage :	`http://ninova.itu.edu.tr/Ders/1318/Sinif/4735`

Tentative Course Outline

- Probability Space

  Probability models, conditioning, Bayes' rule, independence.

- Discrete Random Variables

  Probability mass function, functions of random variables, expectation, joint PMFs, conditioning, independence.

- General Random Variables

  Probability distribution function, cumulative distribution function, continuous Bayes' rule, correlation, conditional expectation.

- Limit Theorems

  Law of large numbers, central limit theorem.

- Introduction to Statistics

  Parameter estimation, linear regression, hypothesis testing.

MAT 271E – Homework 1

Due 22.02.2012

1. You have 4 marbles in your pocket and you try to hit a target with these marbles. Each time you throw, you either hit or miss. If you hit twice, you stop. Otherwise, you continue throwing until you run out of marbles (you do not pick up the marbles thrown earlier). Propose a sample space for this 'experiment'.

2. In a class, 10% of the students wear glasses but don't wear a ring, 20% wear a ring but don't wear glasses and 40% wear neither rings nor glasses. Compute the probability that a randomly chosen student

   (a) wears a ring <u>or</u> glasses,

   (b) wears glasses <u>and</u> a ring,

   (c) wears a ring.

3. A fair die is rolled twice and we assume that all thirty-six possible outcomes are equally likely. Let X and Y be the result of the 1st and the 2nd roll, respectively. Let A, B be events defined as,

$$A = \{X + Y \leq 4\}, \qquad B = \{\max(X, Y) \leq 2\}.$$

   (a) Propose a sample space for this experiment and specify the events A, B, in terms of the sample space.

   (b) Compute $\mathbf{P}(A|B)$.

   (c) Compute $\mathbf{P}(B|A)$.

   (d) Compute $\mathbf{P}(A^c|B)$.

   (e) Compute $\mathbf{P}(B^c|A)$.

4. You and your opponent each roll a fair die <u>with four faces</u>. The one who rolls higher, wins. If the two rolls are equal then it's a draw.

   (a) Propose a sample space for this experiment and specify the event 'you won', in terms of the sample space.

   (b) Suppose you won. What is the probability that your opponent rolled a 3?

   (c) Suppose you lost. What is the probability that your opponent rolled less than 3?

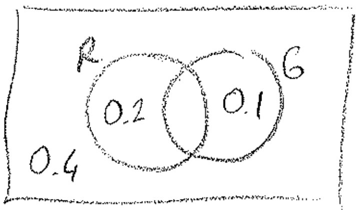
5. We draw four cards from a deck of 52 cards. What is the probability that none of them is an ace? (There are four aces in a deck.)

MAT271E HW1 Solutions

(1) Let H: Hit
M: Miss

$\Omega = \{ \underbrace{HH}_{\text{2 hits}}, \underbrace{MHH}_{\text{1 miss, followed by 2 hits}}, HMH, MMMM, MMMH, MMHM, MMHH, MHMM, MHMH, HMMM, HMMH \}$

(2) Given:



$\Rightarrow P(R \cap G) = 0.3$

(a) $P(R \cup G) = 0.2 + 0.3 + 0.1 = 0.6$

(b) $P(R \cap G) = 0.3$

(c) $P(R) = 0.2 + 0.3 = 0.5$

(3) (a) $\Omega = \{ (1,1), (1,2), \ldots, (1,6)$
$(2,1), \ldots, (2,6)$
$\vdots$
$(6,1), \ldots, (6,6) \}$

$A = \{ (1,1), (1,2), (1,3), (2,1), (2,2), (3,1) \}$
$\Rightarrow P(A) = 6/36$

$B = \{ (1,1), (1,2), (2,1), (2,2) \} \Rightarrow P(B) = 4/36$

(b) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = 1$

(c) $P(B|A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{4/36}{6/36} = \dfrac{2}{3}$

(d) $P(A^c|B) = \dfrac{P(A^c \cap B)}{P(B)} = \dfrac{P(\emptyset)}{P(B)} = 0$

(e) $P(B^c|A) = \dfrac{P(B^c \cap A)}{P(A)} = \dfrac{2/36}{6/36} = \dfrac{1}{3}$

(Notice: $P(A^c|B) = 1 - P(A|B)$
$P(B^c|A) = 1 - P(B|A)$)

4. (a) $\Omega = \{ \quad (1,1), \; \overset{\rightarrow\text{ you}}{(1,}\overset{\rightarrow\text{ your opponent}}{2)}, \; \ldots \; , (1,4)$

$\quad (2,1), \; \ldots \qquad , (2,4) \qquad \Rightarrow 16$ elements

$\quad \vdots$

$\quad (4,1), \; \ldots \qquad , (4,4) \quad \}$

Let $A =$ "You won" $\Rightarrow A = \{ (2,1), (3,1), (3,2), (4,1), (4,2), (4,3) \}$

$$\Rightarrow P(A) = 6/16 \,, \quad P(A^c) = 1 - P(A) = 10/16$$

(b) $B =$ "opponent rolled a 3", $\quad P(B \mid A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{1/16}{6/16} = \dfrac{1}{6}$

$B = \{ (1,3), (2,3), (3,3), (4,3) \}$

(c) $C =$ "opponent rolled less than 3", $D =$ "you lost" $\Rightarrow P(D) \overset{\text{why?}}{=} P(A) = \dfrac{6}{16}$

$C \cap D = \{ (1,2) \} \qquad P(C \mid D) = \dfrac{P(C \cap D)}{P(D)} = \dfrac{1/16}{6/16} = \dfrac{1}{6}$

5. Let $A_1 =$ "First card is not an ace"

$A_2 =$ "Second card is not an ace" $\quad \Rightarrow$ We need to find

$A_3 =$ "Third " " $\qquad P(A_1 \cap \ldots \cap A_4)$

$A_4 =$ " Fourth " "

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_4 \mid A_3 \cap A_2 \cap A_1) \cdot \underbrace{P(A_3 \mid A_2 \cap A_1) \cdot \underbrace{P(A_2 \mid A_1) \cdot P(A_1)}_{= P(A_2 \cap A_1)}}_{\substack{= P(A_3 \cap A_2 \cap A_1) \\ = P(\text{First 3 cards not aces})}}$$

$$= \frac{45}{49} \cdot \frac{46}{50} \cdot \frac{47}{51} \cdot \frac{48}{52}$$

MAT 271E – Homework 2

Due 07.03.2012

1. The king has a sibling. What is the probability that the sibling is male? (Assume that child births are independent of each other and the probability of a boy or a girl is the same.)

2. There are two coins in a bag. One of them has two Heads wheras the other is an unbiased, regular coin. You randomly pick and toss one of the coins and observe 'Heads'.

   (a) What is the probability that the other face is also a 'Head'?

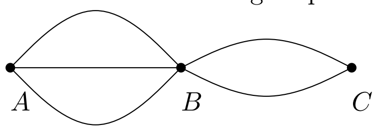
   (b) Without looking at the other face, you toss again. What is the probability that the second toss is also a 'Head'?

3. Let A and B be two events.

   (a) Compute $P(A \cap B^c) + P(A^c \cap B) + P(A \cap B) + P(A^c \cap B^c)$.

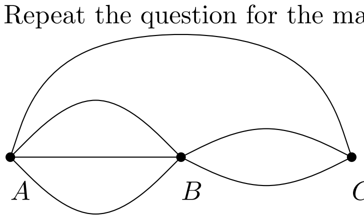
   (b) Suppose that A and B are independent. Show that A^c and B^c are also independent. Hint : Use the total probability theorem and the properties of the probability law.

4. (a) Consider the following map which shows the roads between villages A, B, C.



   Suppose that on a given winter day, a road is blocked with probability p. Assume that the conditions of the roads (i.e. whether they are open or blocked) are independent of each other. Compute the probability that you can go from A to C on a winter day.
   Hint : Think in terms of events like $E_{A\,B}$ = 'I can go from A to B', $E_{B\,C}$ = 'I can go from B to C'.

   (b) Repeat the question for the map below.



5. (a) You roll a fair die until you observe a 6. Assume that the rolls are independent. Let X denote the number of rolls. Write down the PMF of X and compute $\mathbb{E}(X)$.

   (b) Suppose that in the above experiment, you stop if you have rolled 10 times. Let Y denote the number of rolls. Write down the PMF of Y and compute $\mathbb{E}(Y)$.

6. Suppose that the PMF of X is given by

$$p_X(k) = \begin{cases} 1/10, & \text{for } k = -2, \\ 2/10, & \text{for } k = -1, \\ 3/10, & \text{for } k = 0, \\ 4/10, & \text{for } k = 1, \\ 0, & \text{otherwise .} \end{cases}$$

   Let Y be a random variable defined as $Y = X^2$.

   (a) Determine $p_Y(k)$, the PMF of Y.

   (b) Compute $\mathbb{E}(Y)$.

   (c) Compute the variance of X.

MAT271E- HW2 Solutions

(1.) The royal family has two children. The sexes of the children may be $\{BB, BG, GB, GG\}$ where each outcome is equally likely. Given that there is a king, we know that $A=\{BB, BG, GB\}$ has occured. Therefore,

$$P(\{BB\}/A) = \frac{P(\{BB\}\cap A)}{P(A)} = \frac{1}{4} : \frac{1}{3/4} = \frac{1}{3}.$$

(2.) Let $A = \{$Both faces of the picked coin are Heads$\} \Rightarrow P(A)=1/2$.

(a) Let $F_1 = \{$First toss is a Head$\}$.

We need to compute $P(A/F_1)$.

But $$P(A/F_1) = \frac{P(F_1\cap A)}{P(F_1)} = \frac{P(F_1/A)\cdot P(A)}{P(F_1)}.$$

Notice: $P(F_1/A) = 1$, $P(F_1/A^c) = 1/2$.

Also, $$P(F_1) = \underbrace{P(F_1/A)}_{1}\cdot\underbrace{P(A)}_{1/2} + \underbrace{P(F_1/A^c)}_{1/2}\cdot\underbrace{P(A^c)}_{1/2} = \frac{3}{4}.$$

$$\Rightarrow P(A/F_1) = \frac{1\cdot 1/2}{3/4} = \frac{2}{3}.$$

(b) Let $F_2 = \{$Second toss is a Head$\}$.

We are asked to compute $P(F_2/F_1)$

$$P(F_2 \mid F_1) = \frac{P(F_2 \cap F_1)}{P(F_1)}. \quad \text{From (a), } P(F_1) = 3/4$$

$$P(F_2 \cap F_1) = \underbrace{P(F_2 \cap F_1 \mid A)}_{=1} \cdot \underbrace{P(A)}_{1/2} + \underbrace{P(F_2 \cap F_1 \mid A^c)}_{1/4} \cdot \underbrace{P(A^c)}_{1/2} = \frac{5}{8}$$

$$\Rightarrow P(F_2 \mid F_1) = \frac{5/8}{3/4} = 5/6.$$

(Notice: $P(F_2 \mid F_1) > P(F_1)$
How do you interpret this?)

(3.) (a) Rearranging terms, $\underbrace{P(A \cap B) + P(A \cap B^c)}_{= P(A)} + \underbrace{P(A^c \cap B) + P(A^c \cap B^c)}_{P(A^c)} = 1$

(b) We are given that $P(A \cap B) = P(A) \cdot P(B)$. (*)

We are asked to show $P(A^c \cap B^c) = P(A^c) \cdot P(B^c)$.

Let us compute $P(A^c) \cdot P(B^c) = [1 - P(A)][1 - P(B)]$

$$= 1 - P(A) - P(B) + \underbrace{P(A)P(B)}_{= P(A \cap B)}$$

Since $P(B) = P(B \cap A) + P(B \cap A^c)$, we get

$P(A^c)P(B^c) = 1 - P(A) - P(B \cap A^c)$.

From Part (a), we see that the right hand side is equal to $P(A^c \cap B^c)$.

4. (a) We need to compute $P(E_{AB} \cap E_{BC})$.

Since the conditions of the roads are independent, we have

$$P(E_{AB} \cap E_{BC}) = P(E_{AB}) \cdot P(E_{BC}).$$

Notice $E_{AB}^c = \{$all roads from A to B are blocked$\}$.

$$\Rightarrow P(E_{AB}^c) = p^3. \quad \Rightarrow P(E_{AB}) = 1 - p^3.$$

Similarly, $P(E_{BC}) = 1 - p^2$.

$$\Rightarrow P\{I \text{ can go from } A \text{ to } C\} = (1-p^3)(1-p^2)$$

(b) Let $F = \{$The direct path from A to C is open$\}$.

The desired probability is given by

$1 - P(G)$ where $G = \{$cannot reach C from $A\}$.

But $G = \underbrace{[E_{AB} \cap E_{BC}]}_{\text{call this } H}{}^c \cap F^c. \quad \Rightarrow P(G) = P(H^c) \cdot P(F^c)$

$P(F^c) = p \qquad P(H) = (1-p^3)(1-p^2)$

$$\Rightarrow P\{I \text{ can go from } A \text{ to } C\} = 1 - P(H^c) \cdot p$$

5. (a) This is in fact a geometric r.v.

Notice that $X = n$ if the outcome of the rolls is

$$\underbrace{* * * \cdots *}_{\substack{n-1 \text{ rolls} \\ \text{different from } 6}} 6 \Rightarrow P\{X = n\} = \left(\frac{5}{6}\right)^{n-1} \cdot \frac{1}{6}$$

$$\Rightarrow p_X(n) = \begin{cases} \left(\frac{5}{6}\right)^{n-1} \cdot \frac{1}{6}, & \text{if } n \geq 1, \ n \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$$

$$\mathbb{E}(X) = \sum_{n=1}^{\infty} n \cdot \left(\frac{5}{6}\right)^{n-1} \cdot \frac{1}{6}$$

To compute this, notice that $\displaystyle\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} = f(z)$ if $|z| < 1$

Then, $$f'(z) = \sum_{n=1}^{\infty} n z^{n-1} = \frac{1}{(1-z)^2}$$

Therefore (for $z = 5/6$), $$\sum_{n=1}^{\infty} n \cdot \left(\frac{5}{6}\right)^{n-1} = \frac{1}{(1-5/6)^2} = 36$$

$$\Rightarrow \mathbb{E}(X) = \frac{1}{6} \cdot 36 = 6.$$

(You need to roll 6 times on average, to observe a 6).

(b) If $1 \leq n < 10$, then $P\{Y = n\} = P\{X = n\} = \left(\frac{5}{6}\right)^{n-1} \cdot \frac{1}{6}$

But $$P\{Y = 10\} = P\{X \geq 10\} = \sum_{k=10}^{\infty} \left(\frac{5}{6}\right)^{n-1} \cdot \frac{1}{6}$$

$$= \left(\frac{5}{6}\right)^9 \cdot \underbrace{\frac{1}{6} \cdot \sum_{k=0}^{\infty} \left(\frac{5}{6}\right)^k}_{=1} = \left(\frac{5}{6}\right)^9.$$

Therefore $p_Y(n) = \begin{cases} \left(\frac{5}{6}\right)^{n-1} \cdot \frac{1}{6}, & \text{if } 1 \le n \le 10 \\ \left(\frac{5}{6}\right)^{9}, & \text{if } n = 10 \end{cases}$

$$\mathbb{E}(Y) = \sum_{n=1}^{10} n \cdot p_Y(n) = \sum_{n=1}^{9} n \cdot \left(\frac{5}{6}\right)^{n-1} \cdot \frac{1}{6} + 10 \cdot \left(\frac{5}{6}\right)^{9}$$

But if $f(z) = \sum_{n=0}^{N-1} z^n = \frac{1-z^N}{1-z}$,

then $f'(z) = \sum_{n=1}^{N-1} n \cdot z^{n-1} = \frac{-z^{N-1} \cdot N}{1-z} + \frac{1-z^N}{(1-z)^2}$

Setting $z = \frac{5}{6}$, $N = 10$, we have,

$$\frac{1}{6} \sum_{n=1}^{9} n \cdot \left(\frac{5}{6}\right)^{n-1} = -\left(\frac{5}{6}\right)^{9} \cdot 10 + 6 \cdot \left(1 - \left(\frac{5}{6}\right)^{10}\right) = 6 - 15 \cdot \left(\frac{5}{6}\right)^{9}$$

$$\Rightarrow \mathbb{E}(Y) = 6 - 5 \cdot \left(\frac{5}{6}\right)^{9}$$

6. (a) $P\{Y=k\} = p_Y(k) = \begin{cases} \frac{3}{10}, & \text{if } k=0, \\ \frac{2+4}{10}, & \text{if } k=1 \\ \frac{1}{10}, & \text{if } k=4 \\ 0, & \text{otherwise} \end{cases}$

(b) $\mathbb{E}(Y) = 0 \cdot \frac{3}{10} + 1 \cdot \frac{6}{10} + 4 \cdot \frac{1}{10} = 1.$

(c) $\mathbb{E}(X) = -2 \cdot \frac{1}{10} - 1 \cdot \frac{2}{10} + 0 \cdot \frac{3}{10} + 1 \cdot \frac{4}{10} = 0$

$\mathbb{E}(X^2) = \mathbb{E}(Y) = 1$

$\operatorname{var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = 1 - 0 = 1.$

MAT 271E – Homework 3

Due 28.03.2012

1. You cannot remember the last digit of your friend's phone number. You decide to try each digit (in any order you wish). What is the expected number of calls you will make before reaching your friend (without counting your call to your friend)? Assume that the last digits of phone numbers are uniformly distributed (i.e. $P(i) = 1/10$ where i is a digit).

2. You roll a regular unbiased die 10 times. Assume that each roll is independent. Let X be the sum of the rolls. Compute the expected value and variance of X.

3. You have two coins in your pocket. One of them has two heads, the other is a regular unbiased coin. Suppose you randomly pick one of the coins and toss until you observe a 'Head'. Let X be the number of tosses. Compute the mean and variance of X.

4. Let X be a continuous random variable, uniformly distributed on $[0, 1.5]$. We define a discrete random variable Y as,

$$Y = \begin{cases} 1, & \text{if } X - \lfloor X \rfloor < 0.25 \\ 0, & \text{if } X - \lfloor X \rfloor \geq 0.25. \end{cases}$$

   Here, the function $\lfloor X \rfloor$ denotes the integer part of X (i.e. $\lfloor 0.4 \rfloor = 0$, $\lfloor 1.1 \rfloor = 1$ etc.). Find the PMF of Y.

5. You roll a fair die twice. Let X be the greater of the two rolls. Assume that the rolls are independent.

   (a) Find the CDF of X.

   (b) Using part (a), find the PMF of X.

   (Hint : Let X_1 and X_2 denote the outcomes of the first and the second roll. Express X in terms of X_1 and X_2.)

MAT271E - HW3 Solutions

(1.) Let the last digit be 'k'.

In this case, you will make 'k' calls before reaching your friend.

$\Rightarrow$ Let X be the number of calls

$p_X(k) = \frac{1}{10}$ for $k = 0, 1, \ldots, 9$.

$$\Rightarrow \mathbb{E}(X) = \sum_{k=0}^{9} k \cdot \frac{1}{10} = \frac{1}{10} \cdot \frac{9 \cdot 10}{2} = \frac{9}{2}.$$

(2.) Let X_i = outcome of the ith roll.

$$\Rightarrow X = X_1 + X_2 + \ldots + X_{10}.$$

$$\mathbb{E}(X) = \sum_{i=1}^{10} \mathbb{E}(X_i) \quad \text{But } \mathbb{E}(X_i) = \sum_{k=1}^{6} \frac{1}{6} \cdot k = \frac{1}{6} \cdot \frac{6 \cdot 7}{2} = 7/2$$

$$\Rightarrow \mathbb{E}(X) = 35$$

Since the rolls are independent, $\text{var}(X) = \sum_{i=1}^{10} \text{var}(X_i)$.

$$\text{Now } \mathbb{E}(X_i^2) = \sum_{k=1}^{6} \frac{1}{6} \cdot k^2 = \frac{1}{6} \cdot \frac{6 \cdot 7 \cdot 13}{6} = \frac{7 \cdot 13}{6}$$

$$\text{So, } \text{var}(X_i) = \mathbb{E}(X_i^2) - [\mathbb{E}(X_i)]^2 = \frac{7 \cdot 13}{6} - \left(\frac{7}{2}\right)^2 = \frac{7 \cdot (26 - 21)}{12}$$

$$\text{var}(X) = \sum_{i=1}^{10} \text{var}(X_i) = 10 \cdot \frac{7 \cdot 5}{12} = \frac{175}{6}.$$

(3.) Let $A = \{$ You pick the coin with two Heads $\}$.

Then $p_{X|A}(k) = \begin{cases} 1 & \text{if } k = 1 \\ 0, & \text{otherwise} \end{cases} \quad \Rightarrow \mathbb{E}(X|A) = 1$

$$\mathbb{E}(X^2|A) = 1$$

Also, $p_{X|A^c}(k) = \left(\frac{1}{2}\right)^k$, for $k = 1, 2, \ldots$
$(k \in \mathbb{Z}_+)$.

$\mathbb{E}(X|A^c) = \sum_{k=1}^{\infty} k \cdot \left(\frac{1}{2}\right)^k = 2$ (from the prev. homework).

$\mathbb{E}(X^2|A^c) = \sum_{k=1}^{\infty} k^2 \cdot \left(\frac{1}{2}\right)^k = c$

To find c, multiply it by $\frac{1}{2} \Rightarrow \frac{1}{2}c = \sum_{k=1}^{\infty} k^2 \cdot \left(\frac{1}{2}\right)^{k+1}$

$$= \sum_{k=2}^{\infty} (k-1)^2 \left(\frac{1}{2}\right)^k$$

Thus, $c - \frac{1}{2}c = \frac{1}{2} + \sum_{k=2}^{\infty} \underbrace{\left[k^2 - (k-1)^2\right]}_{2k-1} \left(\frac{1}{2}\right)^k$

$$= \frac{1}{2} + \underbrace{2\sum_{k=2}^{\infty} k \cdot \left(\frac{1}{2}\right)^k}_{=2\cdot\left(2-\frac{1}{2}\right)} - \underbrace{\sum_{k=2}^{\infty} \left(\frac{1}{2}\right)^k}_{=\frac{1}{2}}$$

$\Rightarrow \frac{1}{2}c = \frac{1}{2} + 3 - \frac{1}{2} \Rightarrow c = 6.$

Thus, $\mathbb{E}(X) = \mathbb{E}(X|A) \cdot P(A) + \mathbb{E}(X|A^c) \cdot P(A^c)$
$= 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = \frac{3}{2}.$

$\mathbb{E}(X^2) = \mathbb{E}(X^2|A) \cdot P(A) + \mathbb{E}(X^2|A^c) \cdot P(A^c)$
$= 1 \cdot \frac{1}{2} + 6 \cdot \frac{1}{2} = 7/2$

$\Rightarrow \mathrm{var}(X) = \mathbb{E}(X^2) - \left[\mathbb{E}(X)\right]^2 = 7/2 - 9/4 = 5/4.$

(4) $\{Y=1\} = \{0 \le X < 0.25\} \cup \{1 \le X < 1.25\}$

$$P\{Y=1\} = \frac{0.25}{1.5} + \frac{0.25}{1.5} = \frac{1}{3}$$

$$\Rightarrow p_Y(k) = \begin{cases} \frac{1}{3} & \text{if } k=1, \\ \frac{2}{3} & \text{if } k=0. \end{cases}$$

(5.) Let X_1 = result of the first roll

X_2 = result of the second roll

$$\Rightarrow X = \max(X_1, X_2)$$

$$\{X \le k\} = \underbrace{\{X_1 \le k\} \cap \{X_2 \le k\}}_{\text{independent events with probability } \left(\frac{k}{6}\right)}$$

$$\Rightarrow P\{X \le k\} = \left(\frac{k}{6}\right)^2 = F_X(k) \quad \text{for } k \in \{1, 2, .., 6\}.$$

Note: $F_X(k) = \begin{cases} F_X(\lfloor k \rfloor) & \text{for } 1 \le k \le 6 \\ 1 & \text{for } k \ge 6 \\ 0 & \text{for } k < 1 \end{cases}$

(b) $p_X(k) = \begin{cases} F_X(k) - F_X(k-1), & \text{for } k \in \{1, 2, .., 6\} \\ 0, & \text{otherwise} \end{cases}$

$$= \begin{cases} \frac{1}{36}\left[k^2 - (k-1)^2\right] = 2k-1, & \text{for } k \in \{1, 2, .., 6\} \\ 0, & \text{otherwise.} \end{cases}$$

MAT 271E – Homework 4

Due 11.04.2012

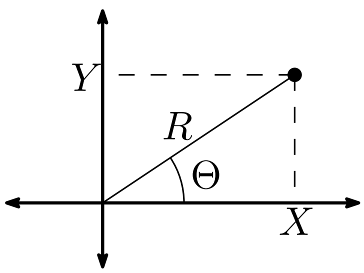
1. Let X, Y be independent random variables, both uniformly distributed on $[0, 1]$.
   (a) Compute the expected value of $|X - Y|$.
   (b) Compute the probability of the event $\{X/2 + 1/2 \geq Y\}$.
   (c) Compute the probability of the event $\{X^2 \geq Y\}$.

2. Let X, Y be random variables with joint pdf given as,

$$f_{X,Y}(t, u) = \begin{cases} 1/\pi, & \text{if } t^2 + u^2 \leq 1, \\ 0, & \text{if } t^2 + u^2 > 1. \end{cases}$$

   (a) Find the marginal pdf of X. Compute $\mathbb{E}(\mathbb{X})$.
   (b) Given that $Y \geq 0$, find the expected value of X – i.e. find $\mathbb{E}(X|Y \geq 0)$.
   (c) Are X and Y independent? Briefly explain your answer.

3. Let X, Y be distributed as in Question-2. We define new random variables R and Θ as $R = \sqrt{X^2 + Y^2}$, $\Theta = \text{atan}(Y/X)$, where 'atan' takes values in the range $[0, 2\pi)$ (think of R and Θ as the polar coordinates – see the figure below).



   (a) Find $F_R(r)$, the cdf of R.
   (b) Derive $f_R(r)$, the pdf of R using part (a).
   (c) Find $F_\Theta(\theta)$, the cdf of Θ.
   (d) Derive $f_\Theta(\theta)$, the pdf of Θ using part (c).
   (e) Find $F_{R,\Theta}(r, \theta)$ the joint cdf of R and Θ.
   (f) Derive $f_{R,\Theta}(r, \theta)$, the joint pdf of R and Θ.
   (g) Are R and Θ independent? Briefly explain your answer.

MAT 271E - HW4 Solutions

(1.) Note that $f_X(t) = f_Y(t) = \begin{cases} 1, & \text{for } t \in [0,1] \\ 0, & \text{for } t \notin [0,1]. \end{cases}$

Thanks to independence, $f_{X,Y}(t,u) = f_X(t) \cdot f_Y(u) = \begin{cases} 1, & \text{for } 0 \le t \le 1, \ 0 \le u \le 1 \\ 0, & \text{otherwise} \end{cases}$

(a) $\mathbb{E}\{|X-Y|\} = \int\int_{-\infty}^{\infty} |t-u| \, f_{X,Y}(t,u) \, dt \, du$

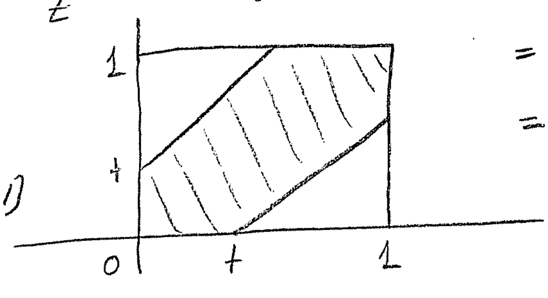
$$= \underbrace{\int_0^1 \int_u^1 (t-u) \cdot dt \, du}_{= \int_0^1 \underbrace{\left(\frac{t^2}{2}\Big|_u^1 - (1-u) \cdot u \right)}_{= \frac{1}{2} + \frac{u^2}{2} - u} du = \frac{1}{2} + \frac{1}{6} - \frac{1}{2} = \frac{1}{6}} + \int_0^1 \int_0^u (u-t) \, dt \, du \quad \Big/ \ = \int_0^1 u \cdot u - \frac{t^2}{2}\Big|_0^u \, du = \int_0^1 \frac{u^2}{2} \, du = \frac{1}{6}$$

$\Rightarrow \mathbb{E}(|X-Y|) = \frac{1}{3}.$

Alternative Solution: Let us define $Z = |X-Y|$ and find $f_Z(t)$. Notice: $F_Z(t) = P\{|X-Y| \le t\} = P\{-t \le X-Y \le t\}$

= Shaded area

$= 1 - (1-t)^2$



$\Rightarrow f_Z(t) = \begin{cases} 2-2t & \text{for } t \in [0,1] \\ 0, & \text{for } t \notin [0,1] \end{cases}$

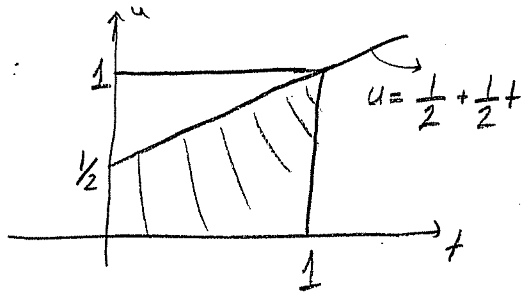
$$\Rightarrow \mathbb{E}(Z) = \int_0^1 t\cdot(2-t)\,dt = \left(t^2 - \frac{2}{3}t^3\right)\Big|_0^1 = 1 - \frac{2}{3} = \frac{1}{3}.$$

(b) $$P\{X/2 + \tfrac{1}{2} \geq Y\} = \int_{-\infty}^{\infty} P\{X/2 + 1/2 \geq t\}\cdot f_Y(t)\,dt$$

$$= \int_0^1 P\{X \geq 2t-1\}\,dt = \int_0^{1/2} \underbrace{P\{X \geq 2t-1\}}_{=1}\,dt + \int_{1/2}^1 \underbrace{P\{X \geq 2t-1\}}_{1-(2t-1)}\,dt$$

$$= \frac{1}{2} + (2t - t^2)\Big|_{1/2}^1 = \frac{1}{2} + \left(2 - 1 - \frac{1}{2} - \frac{1}{4}\right) = \frac{3}{4}.$$

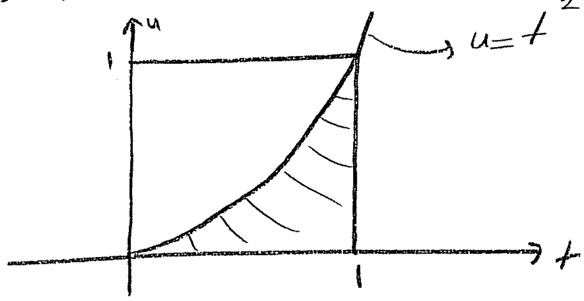
Alternatively, $P\{X/2 + \frac{1}{2} \geq Y\}$ is the shaded area:



(c) $$P\{X^2 \geq Y\} = \int_{-\infty}^{\infty} P\{X^2 \geq t\}\, f_Y(t)\,dt$$

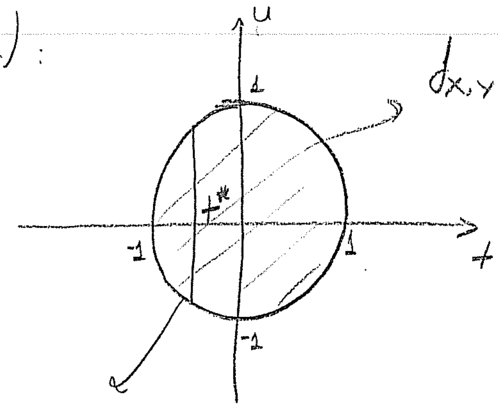
$$= \int_0^1 \underbrace{P\{X^2 \geq t\}}_{=P\{X \geq \sqrt{t}\}}\,dt = \int_0^1 (1-\sqrt{t})\,dt = 1 - \left(\frac{2}{3}t^{3/2}\Big|_0^1\right) = \frac{1}{3}$$

Alternatively, $P\{X^2 \geq Y\}$ is the shaded area:



(2) $f_{X,Y}(t,u)$: $f_{X,Y}(t,u) = \frac{1}{\pi}$ inside the disk.

(a)



$$f_X(t^*) = \underbrace{(\text{length of the segment})}_{=2\cdot\sqrt{1-(t^*)^2}} \cdot \frac{1}{\pi}$$

$$\Rightarrow f_X(t) = \begin{cases} 2\sqrt{1-t^2}, & \text{for } -1 \le t \le 1 \\ 0, & \text{for } t \notin [-1,1]. \end{cases}$$

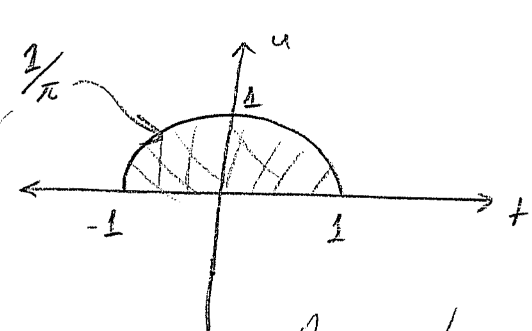
Notice $f_X(t)$ is an even function (i.e. $f_X(t) = f_X(-t)$)

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} \underbrace{t \cdot f_X(t)}_{\text{odd function}}\, dt = 0$$

(b) Note that $P\{Y \ge 0\} = \frac{1}{2}$. Therefore,

$f_{X,Y \mid \{Y \ge 0\}}(t,u) = 2/\pi$

inside the half disk.



Again, by symmetry,

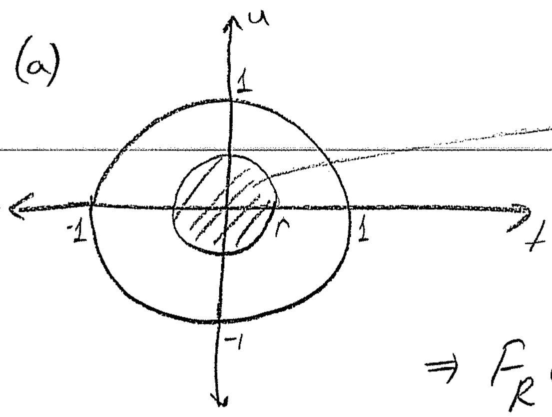
$\mathbb{E}(X \mid Y \ge 0) = 0.$

(c) Similarly as in (a), it can be shown that

$$f_Y(u) = \begin{cases} 2\sqrt{1-u^2}, & \text{for } -1 \le u \le 1 \\ 0, & \text{for } |u| > 1 \end{cases}$$

$\Rightarrow f_{X,Y}(t,u) \neq f_X(t) \cdot f_Y(u)$ so X & Y are not independent.

(3.) (a)



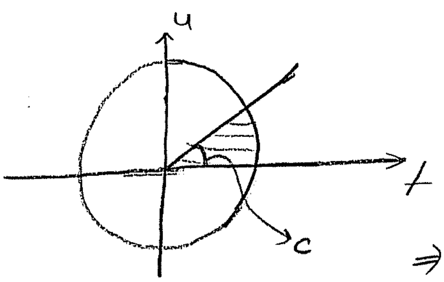
$F_R(r) = P\{R \le r\}$

$\Rightarrow F_R(r) = \frac{1}{\pi} \cdot$ (area of the shaded disk), for $0 < r \le 1$

$$\Rightarrow F_R(r) = \begin{cases} 0, & \text{if } r < 0 \\ 1, & \text{if } r > 1 \\ r^2, & \text{if } 0 \le r \le 1 \end{cases}$$

(b) $$f_R(r) = \frac{d\, F_R(r)}{dr} = \begin{cases} 0, & \text{if } r < 0 \\ 2r, & \text{if } 0 \le r \le 1 \\ 0, & \text{if } r > 1 \end{cases}$$

(c)

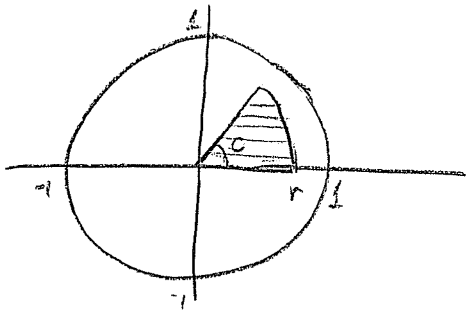


$F_\Theta(c) = \frac{1}{\pi} \cdot$ (shaded area), for $0 \le c < 2\pi$

$$\Rightarrow F_\Theta(c) = \begin{cases} \frac{c}{2\pi}, & \text{for } 0 \le c \le 2\pi \\ 0, & \text{for } c \notin [0, 2\pi]. \end{cases}$$

(d) $f_\Theta(c) = \dfrac{d\,F_\Theta(c)}{dc} = \begin{cases} \frac{1}{2\pi}, & \text{for } 0 \le c \le 2\pi \\ 0, & \text{for } c \notin [0, 2\pi]. \end{cases}$

(e)



$F_{R,\Theta}(r,c) = \frac{1}{\pi}\cdot(\text{shaded area})$ if $\begin{cases} 0 \le r \le 1 \\ 0 \le c \le 2\pi \end{cases}$

$$\Rightarrow F_{R,\Theta}(r,c) = \begin{cases} \frac{cr^2}{2\pi} & \text{if } \begin{array}{l} 0 \le r \le 1 \\ 0 \le c \le 2\pi \end{array} \\ \frac{c}{2\pi}, & \text{if } r > 1 \\ 0, & \text{otherwise} \end{cases}$$

(f) $f_{R,\Theta}(r,c) = \dfrac{d^2 F_{R,\Theta}(r,c)}{dr\,dc} = \begin{cases} \frac{r}{\pi}, & \text{if } \begin{cases} 0 \le r \le 1 \\ 0 \le c \le 2\pi \end{cases} \\ 0, & \text{otherwise} \end{cases}$

(g) $f_{R,\Theta}(r,c) = f_R(r)\cdot f_\Theta(c)$, so R, Θ are independent.

MAT 271E – Homework 5

Due 30.03.2011

1. Let X be uniformly distributed on the interval [-3,2]. We define a discrete random variable Y as,

$$Y = \begin{cases} 1 & \text{if } X \geq 0, \\ 0 & \text{if } X < 0. \end{cases}$$

What is the PMF of Y?

2. You just missed a bus and you're waiting for the next one. Suppose that the probability that the next bus arrives in T units of time (for a non-negative T!) is given by

$$\int_0^T c\, e^{-c\, t}\, dt$$

where 'c' is a constant. Let X be the amount of time you wait for the next bus to arrive.

   (a) What is the probability density function (pdf) of X?

   (b) What is the expected amount of time you need to wait?

   (c) What is the variance of X?

3. The taxis in a city are numbered from 1 to n, where n is the total number of taxis. You try to estimate n as follows. Suppose that after seeing i taxis, your estimate is E_i (and you start from $E_0 = 0$). At your next observation of a taxi, whose number is, say X_{i+1}, you set $E_{i+1} = \max(X_{i+1}, E_i)$. Assume that, any time you observe a taxi, its number is equally likely to be any one in the set $\{1, \ldots, n\}$, independent of previous observations.

   (a) What is the CDF of E_2?

   (b) For $i > 0$, what is the CDF of E_i?

   (c) For $i > 0$, what is the PMF of E_i?

   (d) Compute $\lim_{i\to\infty} \mathbb{E}(E_i)$ and $\lim_{i\to\infty} \operatorname{var}(E_i)$.

MAT271E - HW5 Solutions

1. Y can only take the values $0, 1$.

$$p_Y(0) = P(\{Y=0\}) = P(\{X<0\}) = \int_{-\infty}^{0} f_X(t)\,dt = \int_{-3}^{0} \frac{1}{5}\,dt = \frac{3}{5}$$

$$p_Y(1) = P(\{Y=1\}) = P(\{X\geq 0\}) = \int_{0}^{2} \frac{1}{5}\,dt = \frac{2}{5}.$$

2. (a) Let $s \leq u < 0$. We have that $\underbrace{P\{X\in[s,u]\}}_{=0} = \int_s^u f_X(t)\,dt$

Since $f_X(t)$ is a non-negative function,

this implies that $f_X(t) = 0$ for $t<0$.

Now, if $0\leq s\leq u$,

$$P(\{X\in[s,u]\}) = P(\{X\leq u\}) - P(\{X\leq s\})$$

$$= \int_0^u ce^{-ct}\,dt - \int_0^s ce^{-ct}\,dt = \int_s^u ce^{-ct}\,dt$$

Therefore, by the defn. of pdf, $f_X(t) = \begin{cases} c\,e^{-ct} & \text{for } t\geq 0 \\ 0 & \text{for } t<0 \end{cases}$

(b) $$E(X) = \int_{-\infty}^{\infty} t f_X(t)\,dt = \int_0^{\infty} tce^{-ct}\,dt = -e^{-ct}\,t\Big|_0^{\infty} - \int_0^{\infty} -e^{-ct}\,dt$$

$$= \frac{1}{c}$$

(b) $\operatorname{var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$.

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} t^2 f_X(t)\,dt = \int_0^{\infty} t^2 c e^{-ct}\,dt = -e^{-ct} t^2 \Big|_0^{\infty} - \int_0^{\infty} -e^{-ct} \cdot 2t\,dt$$

$$= \frac{2}{c}\int_0^{\infty} c \cdot e^{-ct} \cdot t\,dt = \frac{2}{c^2} \quad \text{(from part (a))}$$

$\Rightarrow \operatorname{var}(X) = \frac{1}{c^2}$.

3. Notice $p_{X_i}(k) = \frac{1}{n}$ for $1 \le k \le n$. $\Rightarrow P(\{X_i \le k\}) = \frac{k}{n}$.

if k is an integer, and, $1 \le k \le n$.

(a) $E_2 = \max(X_1, X_2)$.

$F_{E_2}(k) = P(\{E_2 \le k\})$. But notice $\{E_2 \le k\}$ is the same event as $\{X_1 \le k\} \cap \{X_2 \le k\}$. Since X_1 and X_2 are independent, we have $F_{E_2}(k) = \frac{k^2}{n^2}$ if k is an integer, and $1 \le k \le n$.

If t is not an integer, define $\lfloor t \rfloor$ = integer part of t.

$$\Rightarrow F_{E_2}(t) = \begin{cases} \frac{\lfloor t \rfloor^2}{n^2} & \text{if } 1 \le t < n+1, \\ 1 & \text{if } t \ge n+1 \\ 0 & \text{if } t < 1 \end{cases}$$

(b) Reasoning similarly as in (a), notice that, $E_i = \max(X_1, X_2, \dots, X_i)$.

$$\{E_i \le t\} = \{X_1 \le t\} \cap \{X_2 \le t\} \cap \dots \cap \{X_i \le t\}.$$

So, $$F_{E_i}(t) = \begin{cases} \dfrac{(\lfloor t \rfloor)^i}{n^i} & \text{if } 1 \le t < n+1 \\ 1 & \text{if } t \ge n+1 \\ 0 & \text{if } t < 1 \end{cases}$$

(c) Since E_i can only take integer values, between 1 and n

$$P(\{E_i = k\}) = P(\{E_i \le k\}) - P(\{E_i \le k-1\}).$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$p_{E_i}(k) = F_{E_i}(k) - F_{E_i}(k-1) = \frac{k^i}{n^i} - \frac{(k-1)^i}{n^i} \quad \text{for } k = 1, \dots, n.$$

(d) $$\lim_{i\to\infty} \mathbb{E}(E_i) = \lim_{i\to\infty} \sum_{k=1}^{n} k \cdot \left\{\left(\frac{k}{n}\right)^i - \left[\frac{(k-1)}{n}\right]^i\right\}$$

$$= \sum_{k=1}^{n} k \cdot \left(\lim_{i\to\infty} \left(\frac{k}{n}\right)^i - \left(\frac{k-1}{n}\right)^i\right) = n$$

Similarly $$\lim_{i\to\infty} \mathbb{E}(E_i^2) = \sum_{k=1}^{n} k^2 \left(\lim_{i\to\infty} \left(\frac{k}{n}\right)^i - \left(\frac{k-1}{n}\right)^i\right) = n^2$$

$$\Rightarrow \operatorname{var}(E_i) = \mathbb{E}(E_i^2) - [\mathbb{E}(E_i)]^2 = 0.$$

MAT 271E – Homework 6

Due 06.04.2011

1. Consider an experiment that involves an infinite number of coins being tossed at the same time. Suppose we number the coins and denote the outcome of the n^{th} toss as O_n. We define

$$X_n = \begin{cases} 1, & \text{if } O_n = \text{Head}, \\ 0, & \text{if } O_n = \text{Tail}. \end{cases}$$

We construct a number (in the base-2 system) as

$$Z = 0.X_1X_2X_3\ldots$$

(a) Propose a sample space, Ω, for this experiment.

(b) Consider the events $A = \{Z \geq 1/2\}$, $B = \{Z \geq 1/4\}$. Express these events in terms of the sample space you proposed in (a).

(c) If the coin tosses are independent of each other and the probability of observing a Head for each coin is given by p, compute the proababilities of A and B.

2. Noise in communication channels at a particular instant is usually modelled as a Gaussian random variable. Let us denote it by X. Suppose for the sake of simplicity that X is a *standard* Gaussian random variable (i.e. it has zero mean and its variance is one). The power of noise is defined as $Y = X^2$. Notice that Y is also a random variable. In this question we will derive the probability distribution function (pdf) of Y. Recall that

$$\Phi(t) = P\big(\{X \leq t\}\big) = \int_{-\infty}^{t} f_X(t)\,dt,$$

where

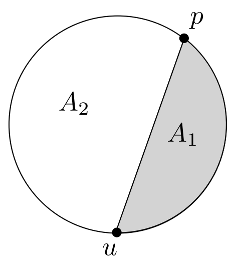
$$f_X(t) = \frac{1}{\sqrt{2\pi}}\,e^{-2t^2}.$$

(a) Express the cumulative distribution function (cdf) of Y, that is $F_Y(s) = P\big(\{Y \leq s\}\big)$ in terms of the function $\Phi(\cdot)$.

(b) Obtain the pdf of Y, denoted by $f_Y(s)$, by differentiating $F_Y(s)$ (you can write it explicitly).

3. You choose two points on a line segment independently, according to a uniform distribution and break the segment into three parts. What is the probability that you can form a triangle with the resulting segments?

4. You randomly choose two points p, u, independently, according to a uniform distribution on a circle of radius 1 as shown below.



The chord between p and u divides the circle into two segments A_1 and A_2, where A_1 is the one with smaller area. Compute the expected areas of A_1 and A_2.

5. Let X_1 be a continuous random variable, uniformly distributed on $[0, 1]$. Let Y be an exponential random variable, i.e.,

$$f_Y(t) = \begin{cases} c\,e^{-ct} & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$

Compute the probability that $X \leq Y$.

MAT271E - HW6 Solutions

1. (a) $\Omega =$ All sequences of the form $(O_1, O_2, \dots)$ where O_i is one of $\{H, T\}$.

(b) $A = (\text{All sequences that start with } H) \cup \{(T, \underbrace{H, H, H, \dots}_{\text{all Heads}})\}$

$B = (\text{All sequences that start that contain at least one } H \text{ in the first two positions}) \cup \{(T, T, \underbrace{H, H, H, \dots}_{\text{all Heads}})\}$

(c) $P(A) = p + (1-p) \cdot p^{\infty} = p$

$P(B) = \left[1 - (1-p)^2\right] + (1-p)^2 p^{\infty} = 1 - (1-p)^2$

2. (a) $F_Y(s) = \begin{cases} P(\{X^2 \le s\}) = P(\{-\sqrt{s} \le X \le \sqrt{s}\}) = \Phi(\sqrt{s}) - \Phi(-\sqrt{s}) & \text{for } s \ge 0 \\ 0 & \text{for } s < 0 \end{cases}$

(b) $f_Y(s) = \begin{cases} \dfrac{\partial}{\partial s} F_Y(s) = \dfrac{1}{2\sqrt{s}} \Phi'(\sqrt{s}) + \dfrac{1}{2\sqrt{s}} \Phi'(-\sqrt{s}) = \dfrac{1}{\sqrt{2\pi s}} e^{-2s} & \text{for } s \ge 0 \\ 0 & \text{for } s < 0 \end{cases}$

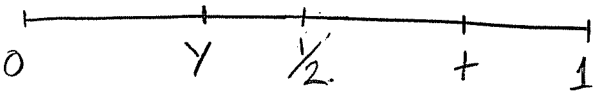
(3) Let X, Y denote the two points.

Let $A_1 = \{X_1 > \frac{1}{2}\}$. $P(A_1) = \frac{1}{2}$.

We remark that, $P(\text{Triangle}) = P(\text{Tri} / A) \cdot P(A) + P(\text{Tri} / A^c) \cdot P(A^c)$.

Also, by symmetry, $P(\text{Tri} / A) = P(\text{Tri} / A^c)$, so $P(\text{Tri}) = P(\text{Tri} / A)$.

Let us compute $P(\text{Tri} / A)$. Notice that given A, X is uniformly distributed on $[\frac{1}{2}, 1]$. $\left(f_{X|A}(t) = \begin{cases} 2 & \text{for } t \in [\frac{1}{2}, 1] \\ 0 & \text{otherwise} \end{cases} \right)$



Assume $X = t$. Then we can form a triangle if and only if

$$\{Y < \tfrac{1}{2}\} \cap \{0 < t - Y < \tfrac{1}{2}\} = \{t - \tfrac{1}{2} < Y < \tfrac{1}{2}\}$$

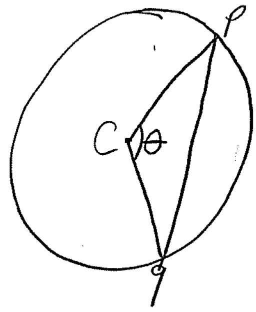
happens. The probability of this event is $= \frac{1}{2} - \left(\frac{1}{2} - t\right) = 1 - t$.

Therefore, $P(\text{Tri} / A) = \int_{1/2}^{1} 2 \cdot (1 - t)\, dt = 1 - \left(t^2 \Big|_{1/2}^{1}\right) = \frac{1}{4}$

$\Rightarrow P(\text{Triangle}) = \frac{1}{4}$.

(See the book for an alternative solution.)

(4) Let C be the center and θ be defined as



θ is uniformly distributed on $[0, \pi]$.

Moreover given θ, Area of $A_1 = \frac{\theta}{2} - \frac{\sin\theta}{2}$ ⟶ Area of triangle

$\Rightarrow$ Expected area of A_1 $= \frac{1}{\pi}\int_0^\pi \left(\frac{\theta}{2} - \frac{\sin\theta}{2}\right) d\theta = \frac{1}{\pi} \cdot \left(\frac{\pi^2}{4} + \frac{\cos\theta}{2}\Big|_0^\pi\right)$

$$\frac{\pi}{4} + \frac{1}{\pi}\left(-\frac{1}{2} - \frac{1}{2}\right) = \frac{\pi}{4} - \frac{1}{\pi}$$

Since (Area of A_2) $= \pi -$ (Area of A_1)

$\mathbb{E}$ (Area of A_2) $= \pi - \left(\frac{\pi}{4} - \frac{1}{\pi}\right) = \frac{3\pi}{4} + \frac{1}{\pi}$

(5) $P(\{Y \geq X\} \mid \{X = s\}) = \int_s^\infty c e^{-ct} dt = -e^{-ct}\Big|_s^\infty = e^{-sc}$

$\Rightarrow P(\{Y \geq X\}) = \int_0^1 e^{-sc} \underbrace{f_X(s)}_{=1} ds = -\frac{1}{c} e^{-sc}\Big|_0^1 = \frac{1}{c}(1 - e^{-c})$

MAT 271E – Homework 7

Due 13.04.2011

1. We are given a biased coin and we are told that the probability of Heads is P. Assume that P is a random variable whose pdf is

$$f_P(t) = \begin{cases} 2t & \text{if } t \in [0,1], \\ 0 & \text{if } t \notin [0,1]. \end{cases}$$

   Suppose we start tossing the coin. Assume that the tosses are independent.

   (a) What is the probability that the first toss is a Head?

   (b) Given that the first toss is a Head, compute the conditional pdf of P.

   (c) Given that the first toss is a Head, compute the probability that the second toss is also a Head.

2. Let X and Y be independent random variables, uniformly distributed on $[0,1]$. Find the pdf of $Z = X/Y$.

3. Let X and Y be independent random variables, uniformly distributed on $[0,1]$. Find the pdf of $Z = X + Y$.

4. Let X and Y be independent random variables, uniformly distributed on $[0,1]$. Find the pdf of $Z = |X - Y|$.

5. Let X be a standard Gaussian random variable. Compute $\mathbb{E}(X^3)$ and $\mathbb{E}(X^4)$.

MAT271E – HW7 Solutions

1 (a) $P(\{H\}) = \int P(\{H\} \mid \{P=t\})\, f_p(t)\,dt = \int_0^1 t \cdot 2t\,dt = \frac{2}{3}.$

(b) Let $A = \{\text{Toss is a Head}\}$. Let $0 \le a \le b \le 1$.

$$P(\{a \le P \le b\} \mid A) = \frac{P(A \cap \{a \le P \le b\})}{P(A)}$$

$$= \frac{\int_a^b (t \cdot 2t)\,dt}{2/3} = \int_a^b 3t^2\,dt = \int_a^b f_{P|A}(t)\,dt$$

$$\Rightarrow f_p(t) = \begin{cases} 3t^2 & \text{for } t \in [0,1] \\ 0 & \text{for } t \notin [0,1] \end{cases}$$

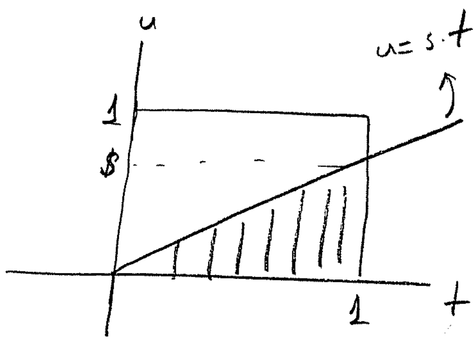
(c) $P(\underbrace{\{\text{Two consecutive Heads}\}}_{B}) = \int P(B \mid \{P=t\}) \cdot f_p(t)\,dt$

$$= \int_0^1 t^2\, 2t\,dt = \frac{2}{4}.$$

$$\Rightarrow P(\{2^{nd}\ \text{Toss} = H\} \mid \{1^{st}\ \text{Toss} = H\}) = \frac{P(B)}{P\{1^{st}\ \text{Toss} = H\}} = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}.$$

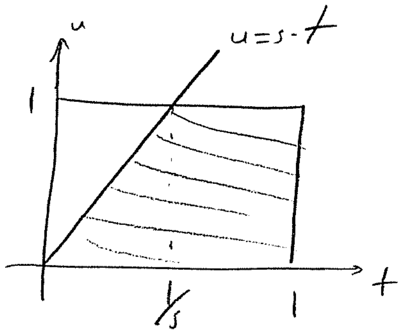
Notice this can also be computed using $f_{P|A}(t)$ as: $\int_0^1 t \cdot 3t^2\,dt = \frac{3}{4}.$

(2.) $f_{X,Y}(t,u) = \begin{cases} 1 & \text{if } 0\le t\le 1 \text{ and } 0\le u\le 1 \\ 0 & \text{otherwise} \end{cases}$



$\underbrace{P(X/Y \le s)}_{\substack{\parallel \\ X \le tY}}$ is equal to the area of the shaded triangle for $s \le 1$

$= \frac{s}{2}$



For $s \ge 1$ $\quad P(X/Y \le s) \Rightarrow$

$= 1 - \frac{1}{2s}$

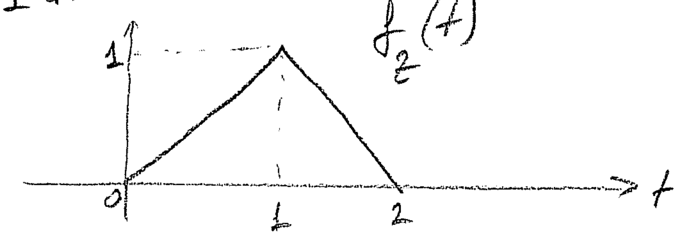
$$\Rightarrow F_Z(s) = \begin{cases} 0 & \text{for } s \le 0 \\ s/2 & \text{for } 0 < s \le 1 \\ 1 - 2/s & \text{for } 1 < s \end{cases} \Rightarrow f_Z(s) = \begin{cases} 0 & \text{for } s \le 0 \\ 1/2 & \text{for } 0 < s \le 1 \\ 2/s^2 & \text{for } 1 < s \end{cases}$$

(3.) $F_Z(s) = P(Z \le s) = \int \underbrace{P(Y + t \le s)}_{= F_Y(s-t)} \cdot f_X(t)\,dt$

$\Rightarrow f_Z(s) = \int f_Y(s-t) f_X(t)\,dt$

If $s \in [0,1] \Rightarrow f_Z(s) = \int_0^s 1 \cdot 1\,dt = s$ $\qquad$ If $s \notin [0,2] \Rightarrow f_Z(s) = 0$

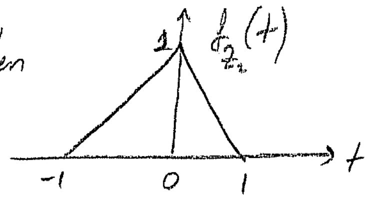
If $s \in [1,2] \Rightarrow f_Z(s) = \int_{s-1}^{1} 1 \cdot 1\,dt = 2 - s$

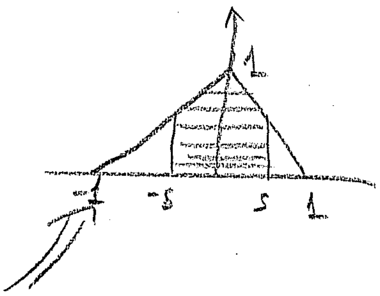


(4.) Let $Z_1 = -Y \Rightarrow f_{Z_1}(t) = \begin{cases} 1 & \text{if } t \in [-1, 0] \\ 0 & \text{otherwise} \end{cases}$

It can be shown, similar to Q3 that if $Z_2 = X + Z_1$

then





Finally $Z = |Z_2|$.

$\Rightarrow$ For $s \in [0, 1]$, $P\{Z \leq s\} = \int_{-s}^{s} f_{Z_2}(t)\,dt = 1 - (1-s)^2$
$= 2s - s^2$

$$\Rightarrow f_Z(s) = \begin{cases} 2 - 2s & \text{for } s \in [0,1] \\ 0 & \text{for } s \notin [0,1] \end{cases}$$

(5.) Recall that the moment generating function for the standard Gaussian is: $M(s) = e^{s^2/2}$;

$$\mathbb{E}(X^3) = \frac{d^3}{ds^3} M(s) \Big|_{s=0}, \quad \mathbb{E}(X^4) = \frac{d^4}{ds^4} M(s) \Big|_{s=0}$$

$M'(s) = s e^{s^2/2}$; $M''(s) = e^{s^2/2} + s^2 e^{s^2/2}$; $M'''(s) = s e^{s^2/2}(1+s^2) + 2s e^{s^2/2}$
$= M(s)(1+s^2)$ $\qquad = e^{s^2/2}(3s + s^3)$

$$\frac{d^4}{ds^4} M(s) = e^{s^2/2}(3s^2 + s^4) + e^{s^2/2}(3 + 3s^2)$$

$\Rightarrow \mathbb{E}(X^3) = 0$ (In fact all odd moments of st. Gaus $= 0$ - Why?

$\Rightarrow \mathbb{E}(X^4) = 3$

MAT 271E – Homework 8

Due 04.05.2011

1. We are given independent, identically distributed (iid) observations $X_1, X_2, \ldots, X_n$ which have Gaussian distributions with mean 3, and unknown variance v.
   (a) Find the maximum likelihood estimate of v.
   (b) Is the estimate you found biased? If it is biased, can you propose an unbiased estimator?
   (c) Is the estimate you found in part (a) consistent?
2. We are given independent, identically distributed (iid) observations $X_1, X_2, \ldots, X_n$ which are uniformly distributed on the interval $[0, \theta]$.
   (a) Find the maximum likelihood estimate of θ.
   (b) Is the estimate you found biased? If it is biased, can you propose an unbiased estimator?
   (c) Is the estimate you found in part (a) consistent?
3. Consider a biased die where the probability of observing a '1' is equal to θ. Let k be a fixed number. You roll the die until you observe k 1's. Let N be the number of rolls. Assuming that the rolls are independent, find the ML estimator of θ based on N.

MAT 271E – Probability and Statistics

Midterm Examination I

14.03.2012

Student Name : ____________________

Student Num. : ____________________

5 Questions, 120 Minutes

Please Show Your Work!

(10 pts) 1. You have a regular, unbiased coin and a regular, unbiased die. Consider the following two step experiment.

(i) You toss the coin.

(ii)
- If the toss is a 'Head', you toss the coin again.
- If the toss is a 'Tail', you roll the die.

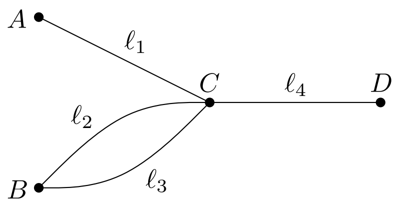
Propose a sample space for this experiment.

(25 pts) 2. You have four coins in your pocket. One of the coins has 'Heads' on both faces. The other three are regular, unbiased coins. You randomly pick one of the coins, toss it and observe a 'Head'. Answer the following questions by taking into account this information.

(a) What is the probability that the other face is also a 'Head'?

(b) Without looking at the other face, you randomly pick one of the remaining coins. What is the probability that this second coin has 'Heads' on both faces?

(25 pts) 3. Consider a network with four nodes (A, B, C, D) and four links (ℓ_1, ℓ_2, ℓ_3, ℓ_4) as shown below.



Assume that at a particular time, a particular link is operational with probability 'p'. Assume also that the conditions of the links (i.e. whether they are operational or not) are independent of each other. We say that two nodes can communicate if there exists at least one path with operational links between them.

(Please provide brief explanations for full credit.)

(a) Compute the probability that A can communicate with D.

(b) Compute the probability that B can communicate with C.

(c) Compute the probability that A and B can both communicate with C.

(d) Compute the probability that A and B can both communicate with D.

(20 pts) 4. You toss an unbiased coin until you observe two 'Heads' (in total) – once you observe two 'Heads', you stop tossing. Assume that each toss is independent of the previous tosses.

(a) What is the probability that you stop after two tosses?

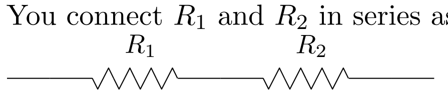
(b) What is the probability that you stop after three tosses?

(c) Let X be the number of tosses. Write down the probability mass function (PMF) of X.

(20 pts) 5. Two bags are handed in to you. There are two resistors in each bag and nothing else. Unfortunately, the bars which indicate the values of the resistors are somehow erased. You are told that in each bag, there is a $1\,\Omega$ resistor and a $2\,\Omega$ resistor.

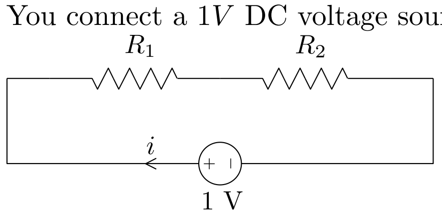
- You randomly pick a resistor from the first bag – let us call this resistor R_1.
- Then, you randomly pick a resistor from the second bag – let us call this resistor R_2.

(a) You connect R_1 and R_2 in series as shown below.



Compute the expected value of the equivalent resistance of this series connection.

(b) You connect a $1V$ DC voltage source to the series connection as shown below.



Compute the expected value of the current i.

MAT 271E – Probability and Statistics

Midterm Examination II

18.04.2012

Student Name : ____________________

Student Num. : ____________________

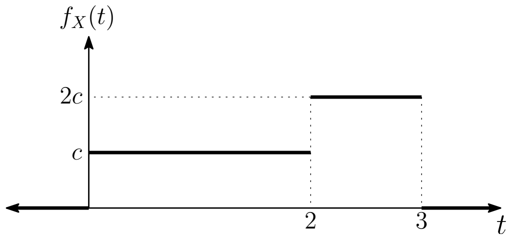
4 Questions, 90 Minutes

Please Briefly Explain Your Answers for Full Credit

(25 pts) 1. Consider a random variable, X, whose probability density function (pdf) is,

$$f_X(t) = \begin{cases} c, & \text{for } 0 \le t < 2, \\ 2c, & \text{for } 2 \le t \le 3, \\ 0, & \text{for } t \notin [0,3], \end{cases}$$

as shown on the right.



Let us define the events $A = \{1 \le X \le 2.5\}$ and $B = \{1.5 \le X \le 3\}$.

(a) Determine c.

(b) Compute $P(A)$, the probability of the event A. (In terms of 'c', if you do not have an answer to part (a).)

(c) Compute $P(B|A)$, the conditional probability of the event B, given A.

(d) Determine and sketch $F_X(t)$, the cumulative distribution function (cdf) of X.

(e) Compute $\mathbb{E}(X)$, the expected value of X.

(25 pts) 2. Let X and Y be independent random variables whose probability density functions are given by

$$f_X(t) = f_Y(t) = \begin{cases} 0, & \text{for } t < 0, \\ e^{-t}, & \text{for } t \ge 0. \end{cases}$$

(a) Compute $p_1 = P\{X \le 2Y\}$ (i.e., the probability of the event $\{X \le 2Y\}$).

(b) Compute $p_2 = P\{X > 2Y\}$.

(c) Compute $p_3 = P\{X \geq Y/2\}$.

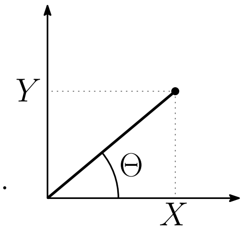
(d) Compute $p_4 = P\{X < Y/2\}$.

(e) Compute $p_5 = P\{Y/2 \leq X \leq 2Y\}$.

(30 pts) 3. Let X and Y be independent random variables, both uniformly distributed on $[0,1]$. Let us define a new random variable Θ as,

$$\Theta = \operatorname{atan}\left(\frac{Y}{X}\right),$$

where the range of 'atan' is restricted to $[0, 2\pi)$ (see the figure).



(a) Determine $F_\Theta(t)$, the cdf of Θ.

(b) Let A be the event $A = \{0 \leq \Theta \leq \pi/4\}$. Determine $F_{X|A}(t)$, the cdf of X given A.

(c) Let A be the event defined in part (b). Compute $\mathbb{E}(X|A)$, the expected value of X given A.

(20 pts) 4. Let X be a random variable whose probability density function is given by

$$f_X(t) = \begin{cases} 0, & \text{for } t < 0, \\ e^{-t}, & \text{for } t \geq 0. \end{cases}$$

Let $Y = 1/X$. Determine $f_Y(t)$, the pdf of Y.

MAT 271E – Probability and Statistics

Final Examination

30.05.2012

5 Questions, 120 Minutes

Please Show Your Work!

(15 pts) 1. There are four balls in a bag. One of the balls is blue and the others are red. You randomly pick a ball, and then, without putting it back, pick a second ball.

(a) Propose a sample space, Ω, for this experiment.

(b) Compute the probability that the second ball is red.

(c) Given that the second ball is red, compute the probability that the first ball is red.

(20 pts) 2. Let X be a random variable whose probability density function (pdf) is given by,

$$f_X(t) = \begin{cases} t/2, & \text{for } t \in [0,2], \\ 0, & \text{for } t \notin [0,2]. \end{cases}$$

Also, let A be the event $A = \{X \geq 1\}$.

(a) Determine $F_X(t) = P\big(\{X \leq t\}\big)$, the cumulative distribution function (cdf) of X.

(b) Compute $\mathbb{E}(X)$, the expected value of X.

(c) Given that A has occured, determine $f_{X|A}(t)$, the conditional probability density function (pdf) of X.

(d) Given that A has occured, compute $\mathbb{E}(X|A)$, the expected value of X.

(25 pts) 3. Let X be a random variable. Suppose we are given the following information regarding its distribution :

$$P\big(\{X \leq t^2\}\big) = 1 - \frac{1}{2}\, e^{-t^2},$$
$$P\big(\{|X| \leq t\}\big) = 1 - \frac{1}{2}\, e^{-t} - \frac{1}{2}\, e^{-2\,t}.$$

(a) Compute the probability of the event $A = \{-1 \leq X \leq 2\}$.

(b) Determine $F_X(t)$, the cumulative distribution function of X.

(c) Determine $f_X(t)$, the probability density function of X.

(15 pts) 4. Let X be a discrete random variable with PMF,

$$P_X(k) = \begin{cases} 1/10, & \text{if } k \in \{1, 2, \ldots, 10\}, \\ 0, & \text{otherwise.} \end{cases}$$

Also, let Y be a random variable with PMF,

$$P_Y(k) = \begin{cases} 1/X, & \text{if } k \in \{1, \dots, X\}, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the probability of the event $\{Y \le 9\}$.

(b) Find the probability of the event $\{Y \le 8\}$.

(25 pts) 5. Let X be a random variable, uniformly distributed on $[0, 1]$. Also, let Z be another random variable defined as $Z = X^\alpha$, where α is an unknown constant.

(a) Compute $f_Z(t)$, the pdf of Z, in terms of α. (You can assume that $\alpha \neq 0$.)

(b) Suppose we are given two independent observations of Z as z_1, z_2. Find the maximum likelihood (ML) estimate of α, in terms of z_1, z_2.

(c) Evaluate the ML estimate you found in part (b) for $z_1 = e^{-3}$, $z_2 = e^{-4}$.